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PRESSURE-DISPLACEMENT COUPLING IN POROELASTICITY. FURTHER DETAILS OF A STABLE FINITE VOLUME FORMULATION

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Abstract. This paper further explores fundamental issues on the behaviour of a finite volume technique using staggered grids for solving poroelasticity problems. Attention is given to the well-known drawback of pressure instabilities, which arises in certain conditions, as in low permeability media, fast transients and undrained conditions. Finite volume techniques are not the first choice for solving poroelasticity problems, and the reason is cultural, since finite elements have a successful history in solving solid mechanics problems. It has been demonstrated, however, that the finite volume strategies can be successfully applied to poroelasticity problems, with the advantage of offering a single method, stable, and fully conservative for both, fluid mass and forces balance.

1 INTRODUCTION

There is a large amount of literature targeting the solution of this coupled problem using finite element for both physics, investigating deeply the reason for the appearance of pressure instabilities, since it is well known that standard Galerkin produces oscillatory pressure solutions, requiring more sophisticated finite element methods, or other strategies to obtain a stable solution (1,2,3], among others. Locking, equal order of interpolation for pressure and displacement, violation of the LBB condition has been claimed as the reason for those instabilities. Despite the origin of instabilities, stabilization techniques need to be devised in order to obtain a solution, and this is a rich field of research among the finite element practitioners, revealing, in the other hand, that the implementation of such stabilizers many times ended up in more complex and time consuming numerical schemes.

It has been shown that finite volume techniques is a viable route for the solution of the coupled fluid flow and geomechanics [4,5,6,7,8,9,10] since the enforcement, at discrete level, of mass conservation for the fluid, and momentum conservation (forces balance) for the rock, renders to the method physical consistency, and, consequently, stability and robustness.

However, even using finite volume methods, stabilization techniques are still required if pressure and displacement are co-located on the computational grid. Fortunately, since these stabilization techniques are founded on physical grounds, it is straightforward to derive them. This matter will be discussed in a following section, demonstrating that if a fully stable solution is devised, with no need of stabilization techniques at all, the use of staggered grids is the answer. This paper recapitulates the main features of a stable finite volume method for poroelasticity employing staggered grids, pointing out the similarities of the pressure-displacement coupling in poroelasticity with the pressure-velocity coupling in the Navier-Stokes equations. Results are shown for the Terzaghi's column using non-uniform grids.

2 THE REASON FOR PRESSURE WIGGLES IN FLUID MECHANICS

In order to have a clearly understanding of the issue to be discussed, let's just point out, by now, that the coupling between pressure and velocity, linking the mass conservations equation with the momentum conservation equations in Navier-Stokes flows, is totally similar to the coupling between pressure and displacement, linking the mass conservation equation with the momentum conservation equations (forces balance) in poroelasticity. Therefore, the remedies for avoiding pressure wiggles in Navier-Stokes flows can be ready extended to poroelasticity problems. The following section briefly discuss the origin of the wiggles in CFD and the remedy already available for more than five decades ago [11].

2.1 Pressure-Velocity Coupling in Navier-Stokes equations

Firstly, it is didactic to have in mind that the coupling we discuss herein is not related with the procedure adopted for solving the linear systems for pressure and displacements, like explicit, iterative or simultaneous (monolithic), but related to the numerical schemes used to create the linear systems, which are dependent on the type of grid used, if co-located or staggered.

Numerical analysts faced the pressure-velocity coupling difficulties in the late 60's, when the stream function-vorticity formulation was abandoned due to its difficulty in dealing with the boundary conditions of fluid flow problems, and the primitive variables, pressure and velocity, took place. At that time, co-located grids, in which pressure and velocity are calculated at the same point, was the standard approach for any numerical scheme for fluid flow. This arrangement, however, give rise to the well-know checkerboard pressure field, situation in which a spurious decoupled pressure field may appear in the solution, as explained in [12]. There are several concurrent reasons why co-located grid creates the possibility of pressure instabilities.

Consider Figure. 1, in which a 1D grid is shown for the solution of a 1D Navier-Stokes problem using co-located variables, i.e., using the same control volume for mass and momentum conservation. Two mains difficulties arise when using this grid layout.

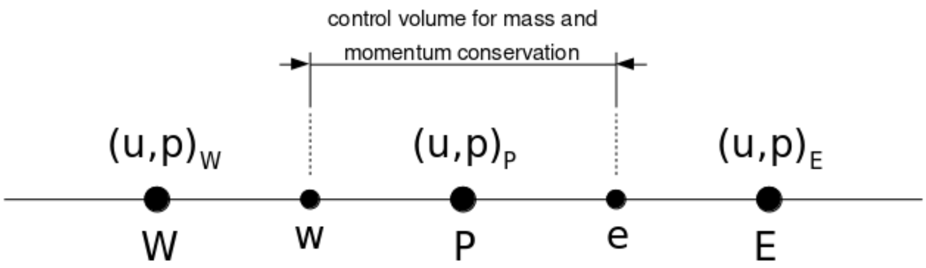

Figure 1: Colocated grid for 1D problem.

Firstly, the integration of the momentum conservation equation will require pressure values at the interfaces of the control volume, points "e" and "w". Since pressure variables are not stored on those points, one must count with values located at points "W", "P" and "E", resulting for the pressure gradient evaluation for the control volume centred at point P, $\left(\frac{P_E - P_W}{2\Delta x}\right)$, disappearing the pressure value at the control volume in consideration, point P, and having the pressure gradient evaluated in a coarser grid [12]. Secondly, and the strong reason why decoupled pressure fields and wiggles appear, is because momentum and mass conservation are not satisfied for the same set o velocities. When integrating the mass conservation equation, velocities at the points "e" and "w" are required to construct the approximate equation obeying the mass balance at the control volume centred at P. However, they are not available at those locations, requiring an extrapolation from the nodal values. This extrapolation is a key issue and can be done in several ways, and are, in fact, forms a family of stabilization schemes employed when solving Navier-Stokes equations using co-located variables [13,14,15]. The stability will be attained depending on the fidelity that these interface velocity represents the physics of the phenomena. If, for example, u_e is taken as a linear interpolation among u_P and u_E, it will be a very poor approximation, since this averaging will consider that the physics from the point "P" to point "e" is purely diffusive, generating a scheme with pressure instabilities. Therefore, to have good stabilizing schemes, the physics between these two points must be well represented.

2.2 The remedy for the pressure wiggles and instabilities in CFD

As mentioned, Harlow and Welch, in a 1965 paper, solving free surface flows using the MAC (mark and cell) method, advanced the staggered layout of variables, as shown in Figure 2. In this case control volumes for mass and momentum conservation are no longer coincident, eliminating all difficulties, since now pressures are available at the interface of a momentum control volume for a proper evaluation of the pressure gradient, and velocities are available at the interfaces of a mass conservation control volume for mass balance. It should be observed that the important fact now is that the same set of velocities satisfying mass conservation also satisfies momentum conservation. Recalling that when the grid is co-located the specification of a physically consistent velocity at the interface of a mass balance control volume is the basis of a stabilizing scheme, it becomes clear that when the proper unknown velocity is moved to the interface, this is the best stabilizing scheme, since the physics is represented with the highest possible fidelity. And transferring the node velocity from the center to the interface of a mass control volume is precisely the staggered grid.

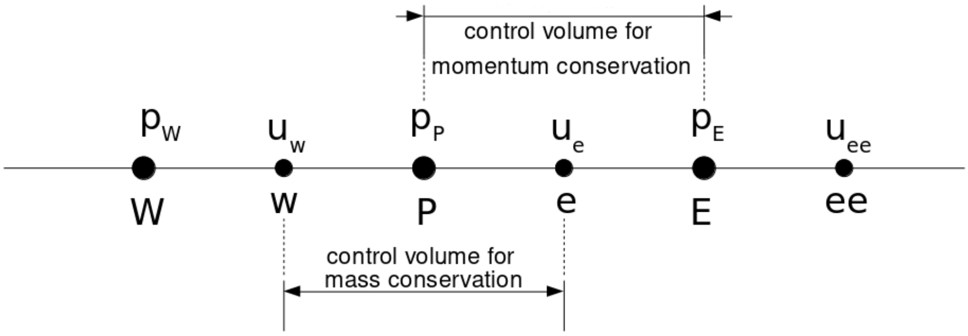

Figure 2: Staggered grid for 1D situation

All schemes that find a way of better representing the nodal velocity at the interface form the family of stabilizing schemes, being the scheme that puts the nodal velocity at the interface (staggered grid) the best one. Therefore, there is no better stabilization scheme than using staggered grids. In the following section the mathematical model for the poroelasticity problem is presented, highlighting the similarities among the pressure-velocity coupling in Navier-Stokes equations and the pressure-displacement coupling in poroelasticity.

3 MATHEMATICAL MODEL FOR THE COUPLED PROBLEM

The solution of a poroelasticity problem comprises in solving the rock mechanics (momentum conservation) of the porous structure coupled with the fluid flow (mass conservation) in the porous structure. Terzaghi introduced the principle of effective stress, stating that the mechanical behaviour of the rock depends on its mechanical properties as well as of the pressure caused by the fluid flowing in the porous space. This principle is written as

$$\nabla.\sigma - \alpha \nabla p = \mathbf{b} \tag{1}$$

in which σ is the effective stress tensor, α is the Biot coefficient, p the pressure and $\mathbf{b}$ stands for a possible source term. Recall that this equation, when solved, furnish the porous media displacement caused by loads and the fluid pressure. The porous pressure must be determined using the mass conservation equation, written in a convenient form for our purposes, by

$$\frac{1}{M}\frac{\partial p}{\partial t} + \nabla \cdot \left(\mathbf{v}^f + \alpha \mathbf{v}^s \right) = q \tag{2}$$

with $\frac{1}{M}$ being the Biot module, q is a source term, and $\mathbf{v}^f$ and $\mathbf{v}^s$, are, respectively, the velocity of the fluid (Darcy's equation) and of the solid, given by

$$\mathbf{v}^f = -\frac{\mathbf{k}}{\mu} \cdot \nabla p \tag{3}$$

and

$$\mathbf{v}^s = \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \varepsilon}{\partial t}$$

Recall that in Section 2, $\mathbf{u}$ was used for denoting the velocity vector in Navier-Stokes flows. From Section 3 forwarded, it stands for displacement of the porous media.

3.1 Coupling similarities – Navier-Stokes flows and Poroelasticity

One of the challenging aspects in solving the poroelasticity equations is to avoid pressure oscillations which appear under certain conditions resembling undrained consolidation, as in the very beginning of transients and at the interface of two media with large difference in permeability. Finite element practitioners devote a large amount of research efforts in creating stabilizing schemes to avoid those pressure wiggles, normally derived based on mathematical background, lacking, many times, the corresponding physical insight.

The characterization of an undrained consolidation process is when, by some reason, the fluid in the porous space is not flowing, situation in which Eq. (2) can be written as,

$$\frac{1}{M}\frac{\partial p}{\partial t} + \nabla \cdot \alpha \mathbf{v}^s = q \tag{5}$$

which exhibits exactly the same form as the mass conservation equation for Navier-Stokes flows, just replacing $\mathbf{v}^s$ by the fluid velocity. It is clear, therefore, that the pressure wiggles encountered in solving Navier-Stokes flows are also present when solving poroelasticity problems under undrained conditions. Hence, the remedies are the same ones. If staggered grids are used, the stabilization is intrinsically applied since the displacement is located at the boundaries of the control volume and plays the same role as velocity in Navier-Stokes flows. Displacements in time are, in fact, solid velocity, and it plays the role of fluid velocity, as seen in Eq. (5). If co-located grids are used, finite volume techniques create stabilization schemes based on physical process, as will be described in the following section.

4 NUMERICAL APPROXIMATION

4.1 Staggered grid

In this section it will reported the numerical approximation for a 1D poroelasticity problem, highlighting the details commented previously. The finite volume approach is used, whereby the conservation equation is integrated over each control volume that subdivides the full domain. In the case under analysis the control volume for mass conservation is 1D with interfaces denoted by "w" and "e", according to Figure 2. Interested readers may find all the numerical developments using staggered grids for 2D problems in the context of Navier-Stokes equations with hybrid unstructured grids in [16]. Writing Eq. (2) replacing the divergence of the solid velocity by the variation in time of the displacements, and using the Darcy's equation for the fluid velocity, one has

$$\left(\frac{1}{M}\right)\frac{\partial p}{\partial t}-\nabla.\left(\frac{k}{\mu}\nabla p\right)=q-\alpha\frac{\partial \varepsilon}{\partial t} \tag{6}$$

Considering the 1D situation in analysis, the integration of the above equation on the grid show in Figure 2, yields,

$$\frac{\Delta x}{M}\frac{p_p}{\Delta t}-\frac{k}{\mu}\left[\left.\frac{\partial p}{\partial x}\right|_e-\left.\frac{\partial p}{\partial x}\right|_w\right]+\frac{\alpha}{\Delta t}\left(u_e-u_w\right)=q_p\Delta x+\frac{\Delta x}{M}\frac{p_p^o}{\Delta t}+\frac{\alpha}{\Delta t}\left(u_e-u_w\right)^o \tag{7}$$

Inspecting Eq.(7) against the grid layout of Figure 2, one sees that the displacements are available were they are required for the calculation of the solid velocity, the key issue for stability in undrained conditions. The pressure gradients are calculated by

$$\left.\frac{\partial p}{\partial x}\right|_e=\frac{(p_E-p_P)}{\Delta x} \tag{8}$$

$$\left.\frac{\partial p}{\partial x}\right|_w=\frac{(p_P-p_W)}{\Delta x} \tag{9}$$

The pressure gradients in the above equations are the responsible for driving the Darcy's velocity and for unstructured grids a gradient recovery method should be employed. Recall that in all equations reported a constant Δx is used. When non-uniform grids are employed, as in one of the results to be presented, the proper local dimension should be used. Substituting Eqs. (8) and (9) into Eq. (7) one gets the linear system to be solved for the pressure determination.

To complete the integration procedure, the momentum conservation equation, Eq. (1), should be integrated in time and space, using the grid shown in Figure 2. Considering the control volume for displacement centred in "e", neglecting the source term, the integration gives,

$$\sigma_E-\sigma_P=\alpha(p_E-p_P) \tag{10}$$

in which one can see that the driving force for the displacement is correctly stored at the grid, another factor contributing for the stability of the scheme. The expressions for the stress tensor, for a 1D case, are given by

$$\sigma_E = \left(\lambda + 2G\right)\frac{\partial u}{\partial x}\bigg|_E \approx \left(\lambda + 2G\right)\frac{(u_{ee} - u_e)}{\Delta x} \tag{11}$$

$$\sigma_P = \left(\lambda + 2G\right)\frac{\partial u}{\partial x}\bigg|_P \approx \left(\lambda + 2G\right)\frac{(u_e - u_w)}{\Delta x} \tag{12}$$

Introducing the above equations into Eq. (10), one gets a linear system for the determination of the u displacement with the fluid pressure present in the equation. Therefore, one has two linear systems, one for pressure and other one for displacement, which can be solved using the strategies already mentioned in this paper. Commenting on these strategies is not in the scope of this paper.

4.2 Co-located grid

Considering now the co-located grid of Figure 1, the control volumes for the integration of the mass and momentum conservation equations are the same. To clarify the difficulties with the mass conservation equation it is repeated,

$$\frac{\Delta x}{M}\frac{p_p}{\Delta t} - \frac{k}{\mu}\left[\frac{\partial p}{\partial x}\bigg|_e - \frac{\partial p}{\partial x}\bigg|_w\right] + \frac{\alpha}{\Delta t}\left(u_e - u_w\right) = q_p \Delta x + \frac{\Delta x}{M}\frac{p_p^o}{\Delta t} + \frac{\alpha}{\Delta t}\left(u_e - u_w\right)^o \tag{13}$$

with the marked difference that now the displacements required at points "e" and "w", the interfaces of the control volume for mass conservation, are not available, requiring, therefore, some stabilization scheme, as discussed in the previous section. In this paper no stabilization schemes are used in conjunction with co-located variables, since the idea is to show the appearance of pressure instabilities.

4.3 Error approximation

The evaluation of the pressure and displacement gradients is made by finite differences with 2nd order errors when the grid is uniform, and somewhere in between 1st and 2nd order when the grid is non-uniform. Therefore, few questions arise concerning the connexion between the order of approximation and stability, as

1) Are pressure and displacements 2nd order accurate?
2) Does pressure and displacement have the same order of accuracy?
3) Will the staggered grid mitigate the numerical instabilities, even in presence of low accuracy?

The following results tries to put some lights over these questions by solving a very simple 1D problem, the Terzaghi's column using both types of grids.

5 RESULTS

The 1D Terzaghi's problem is solved using staggered grids with equally and unequally grid sizes. Consider a domain with length L divided into n subdomains with equally spaced grids, $\Delta x = \frac{L}{n}$, resulting in (n+1) grid points. The coordinates for the equally spaced grids are given by

$$x_i = x_1 + (i-1)\Delta x, \qquad i = 1 \ldots n+1 \tag{14}$$

in which x_1 is the coordinate of the first grid point, normally made equal to zero. The coordinate $\bar{x}$ of the grid points in the non-uniform grid is calculated based on the equally spaced grid, by

$$\bar{x}_i = x_i + \theta\beta\Delta x, \qquad i = 1 \ldots n+1 \tag{15}$$

with β being a number between 0 and 1, and θ the random parameter that controls how much distortion the grid will be subjected to. By the way, $\theta = 0$ recovers the equally spaced grid and $\theta = 0.9$ generates a highly non-uniform grid, as can be seen in Figure 3.

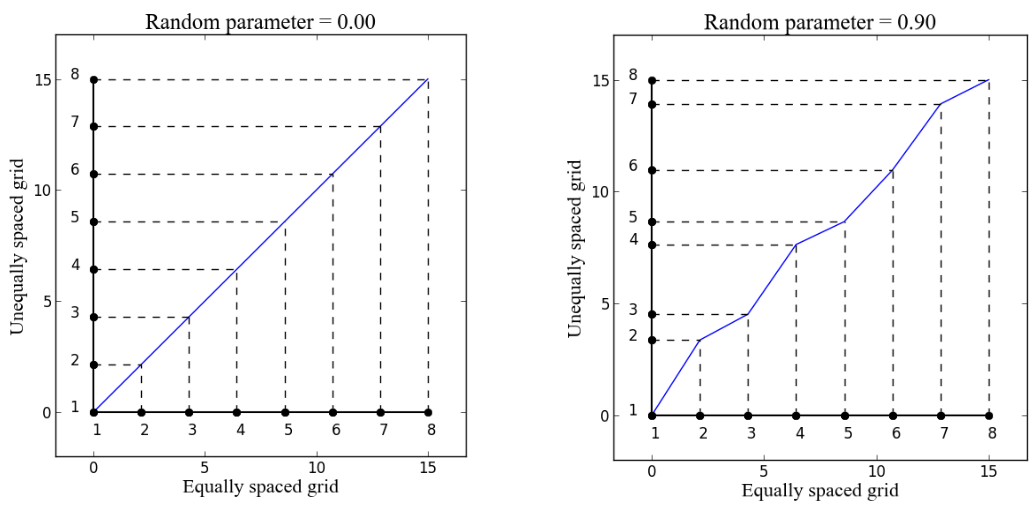

Figure 3: Generation of a non-uniform grid based on equally spaced intervals

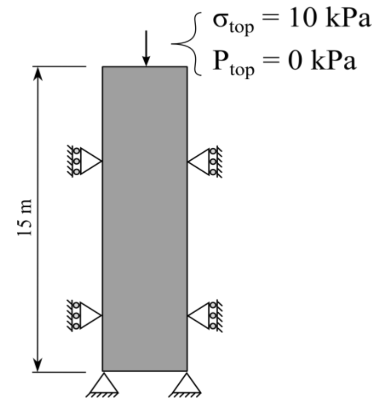

Figure 4: Geometry and boundary conditions for the 1D Terzaghi's problem

As depicted in Figure 4, the solution domain has its bottom boundary fixed and impermeable and the top boundary is fully-permeable ($p_{top} = 0$ kPa) and subjected to a compressive load of $\sigma_{top} = 10$ kPa. The structure is initially non-deformed and the initial pore pressure equals to zero. The fluid phase properties are: $\rho = 998{,}2\ \mathrm{kg/m^3}$, $\mu = 1{,}002\times10^{-3}$ Pa.s and $c_f = 1{,}0\times10^{-4}\ \mathrm{MPa^{-1}}$. The solid phase properties are: $G = 1{,}732$ MPa, $\lambda = 2{,}597$ MPa, $\phi = 0{,}3$, $\alpha = 1{,}0$ and $K = 1{,}0\times10^{-4}$ m/s, where K represents the hydraulic conductivity.

To begin presenting the results, a validation is done comparing the numerical and analytical solutions, as shown in Figures 5 for highly distorted grid. The results for $\theta = 0.1$ are not shown for lack of space. As can be seen, the numerical results agree very well with the corresponding analytical ones for highly non-uniform grid.

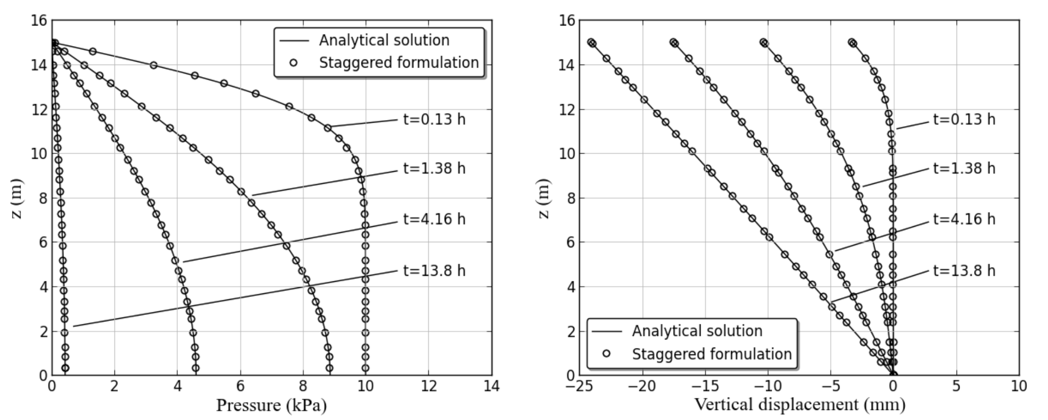

Figure 5: Pressure and displacement fields for θ=0.9

Concerning the convergence rate, two sets of grids were used. The first and second sets consider randomly spaced grids with $\theta = 0,1$ and $\theta = 0,9$, respectively. For each set of grids, pressure and displacement profiles are taken at $t = 500$ seconds. These profiles are compared with the analytical solutions and the Euclidean norm (L_2-norm) of the error vector is computed. Four different time step sizes are considered: 0,1, 1, 10 and 100 seconds.

The behavior of the pressure and displacement error as the grid is refined is presented in Figure 6 for randomly spaced grids with $\theta = 0,9$. As can be verified, a second order decay of the error is obtained and a 2nd order approximation is clearly obtained for the displacement. For pressure, however, the figure suggests that this variable is somehow affected by the grid distortion, but it can still be regarded as a second order approximation.

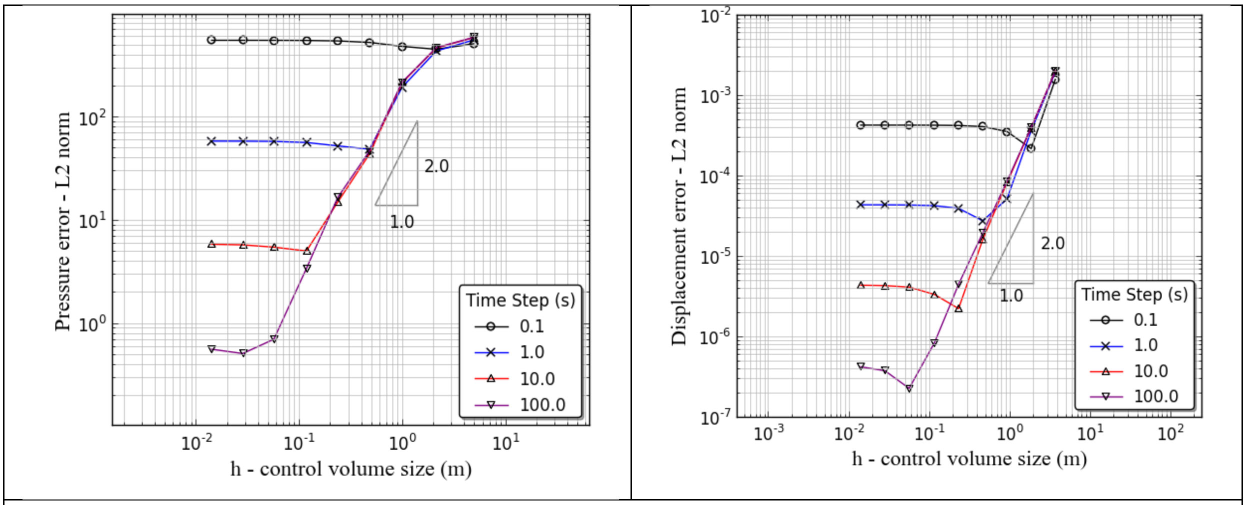

Figure 6: Convergence analysis for the grid $\theta = 0,9$. (a) Pressure , and (b) displacement convergence

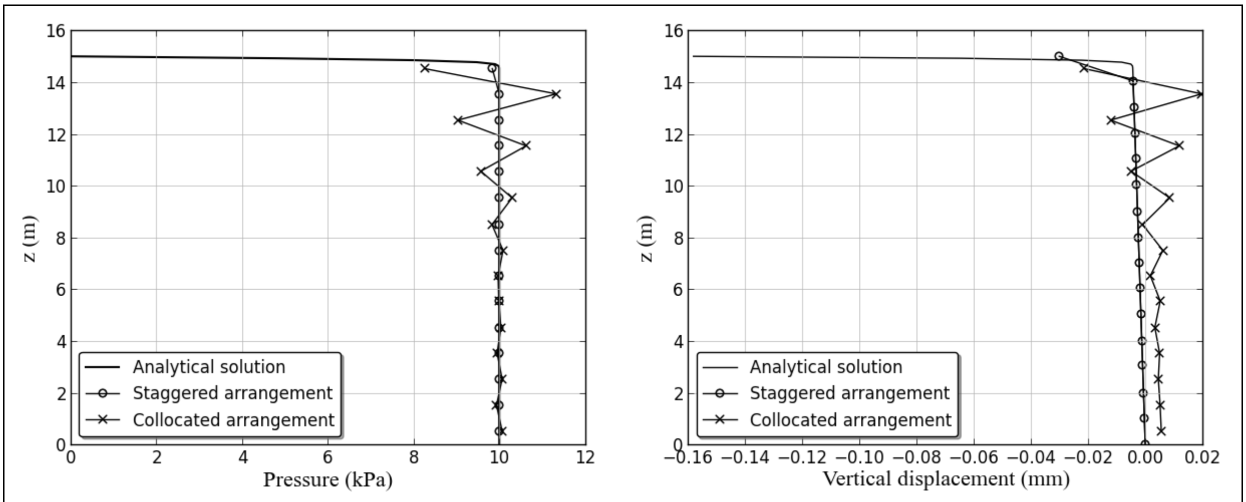

Figure 7: Analytical, staggered and co-located grid results for $\theta = 0.1$ - 16 nodes

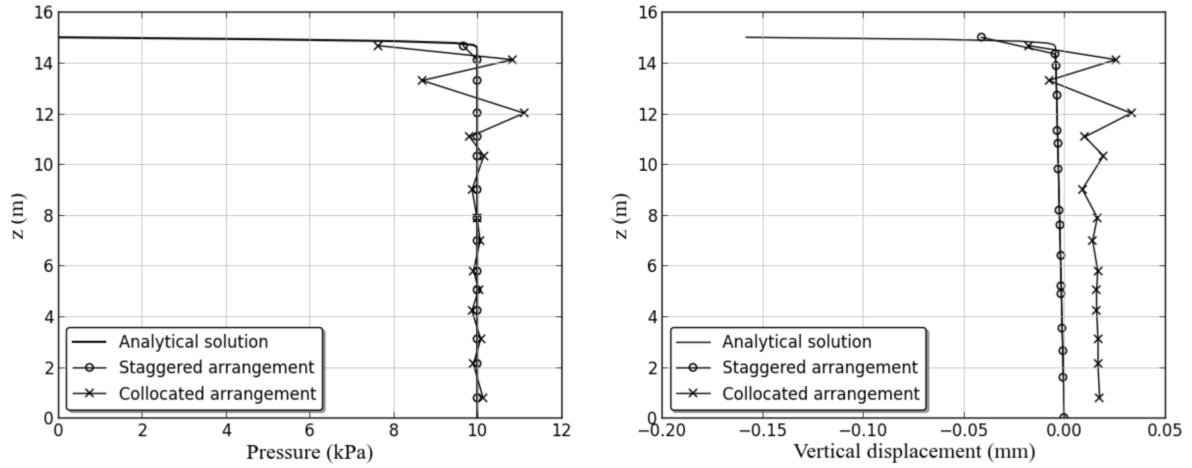

Figure 8: Analytical, staggered and co-located grid results – $\theta = 0.9$ - 16 nodes

Figures 7 and 8 show results for the Terzaghi's column using very small time steps in the beginning of the computations creating conditions for the appearance of instabilities. The co-located grid results are, of course, without stabilization schemes, since tests already performed in other works demonstrate that the use of the PIS scheme in conjunction with co-located grids eliminates the pressure wiggles [14,7]. It is well known that non-uniform grids deteriorate the order of the finite difference approximations when the non-uniformity increases, but in this problem, even using highly non-uniform grids, it was demonstrated that the stability is not influenced by the grid non-uniformity.

6 CONCLUSIONS

This paper presented a discussion about the stabilization schemes required for avoiding pressure wiggles and instabilities in poroelasticity problems when the flow conditions resemble an undrained situation. The similarities among the pressure-velocity coupling in Navier-Stokes flows and pressure-displacement coupling in poroelasticity is clearly stated, and it is shown that when co-located grids are used, despite the method used, stabilization schemes are required. It is also pointed out that when the stabilization algorithm is seen under a physical perspective, the quality of the schemes is dependent on the fidelity in which the interface displacement is related to the nodal point displacement. Few results were shown in which it is possible to verify that the use of staggered grids contains intrinsically the best stabilization scheme.

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