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ADDITIVE CORRECTION MULTIGRID METHOD APPLIED TO DIFUSION PROBLEMS WITH UNSTRUCTURED GRIDS

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Abstract. *This paper shows the potentialities of the Additive Correction Multigrid method (ACM) for solving linear systems of algebraic equations. The motivation of multigrid methods is to solve efficiently a large set of algebraic equations resulting from the discrete approximation of differential equations. ACM is used to improve the convergence rate of iterative methods using several meshes and reducing all frequency modes of the error. In this method, the discretization is made only in the fine grid, reducing the complexity and the computational cost, as well as eliminating the possibility of inconsistencies among grids. Other advantages can be mentioned: enforcement of conservation over all grids by requiring that the sum of the fine grid equations within each coarse block be satisfied; boundary conditions, once incorporated in the fine grid equations require no more considerations, and ACM method doesn't have any difficulties with anisotropic coefficients, because it allows the use of an adaptive agglomeration cell scheme. The ACM method can be applied to any type of mesh, structured or not, which points out its flexibility. To exemplify the robustness of ACM method in practical situations some heat conduction problems were chosen and their equations were discretized using the Element-based Finite Volume Method (EbFVM) in quadrilaterals unstructured grids.*

Keywords. *Additive Correction Multigrid, Element-based Finite Volume Method, unstructured grids, convergence acceleration.*

1. Introduction

Multigrid methods solve efficiently large sets of algebraic equations through the acceleration of iterative methods. Their motivation is to solve the equations in many grids reducing all frequency modes of the error. On the finest (original) grid, the high-frequency modes of the error are effectively reduced, but the low-frequency modes are difficult to remove, since they would require coarser grids to attain this goal. With a minimal computational effort, multigrid methods can eliminate low-frequency modes of the error solving the coarse grid error approximation, which is less expensive than solve the original equations. In the Additive Correction Multigrid method (Hutchinson and Raithby, 1986), simply called here as ACM, the discretization is carried out only in the finest grid, and the coarse grid equations are generated by the summing the finer grid equations. Hence, the conservation is achieved also in the coarse grids.

Studies of iterative methods have shown that their performance decreases when the coefficients of the linear equation become highly anisotropic (Elias et al., 1997). This anisotropy of coefficients causes different timescales for propagation of information, which dramatically affect the convergence behavior. Thus, a kind of adaptive agglomeration, which is provided by the ACM method, turns the multigrid method more efficient because the variation in timescales for propagation of information on a coarse-grid cell is reduced. The procedure consists in adding the fine-grid cells with small transport timescales.

Concerning the types of multigrid, the literature shows that the full multigrid technique (FMG) is the most efficient version (Trottenberg, 2001), being this is the reason why it is used in all examples in this paper. The FMG principle is that the nested iteration idea uses coarse grids to obtain improved initial approximation for the fine grid problems.

In order to evaluate the potentialities of the ACM method, in this article are analyzed some heat conduction problems, whose differential partial equations are discretized using the Element-based Finite Volume Method (EbFVM). This technique is proposed to combine the good features of both, the Finite Volume Method (FVM) and Finite Element Method (FEM). In the former, the balances are made in control volumes satisfying locally and globally the conservation principles, while, in the last, it is used a local system coordinate with fully independency among grid elements. Nowadays, the irregular meshes (unstructured), like the ones used in the EbFVM, are mostly used due to their great flexibility for discretizing complex domains. Besides, the process of automatic generation of unstructured-grid has became simpler.

Initially, this paper outlines the ACM method and its adaptive agglomeration scheme. Then, some considerations about the EbFVM are made. Finally, some examples are included to demonstrate the potentialities of the proposed method.

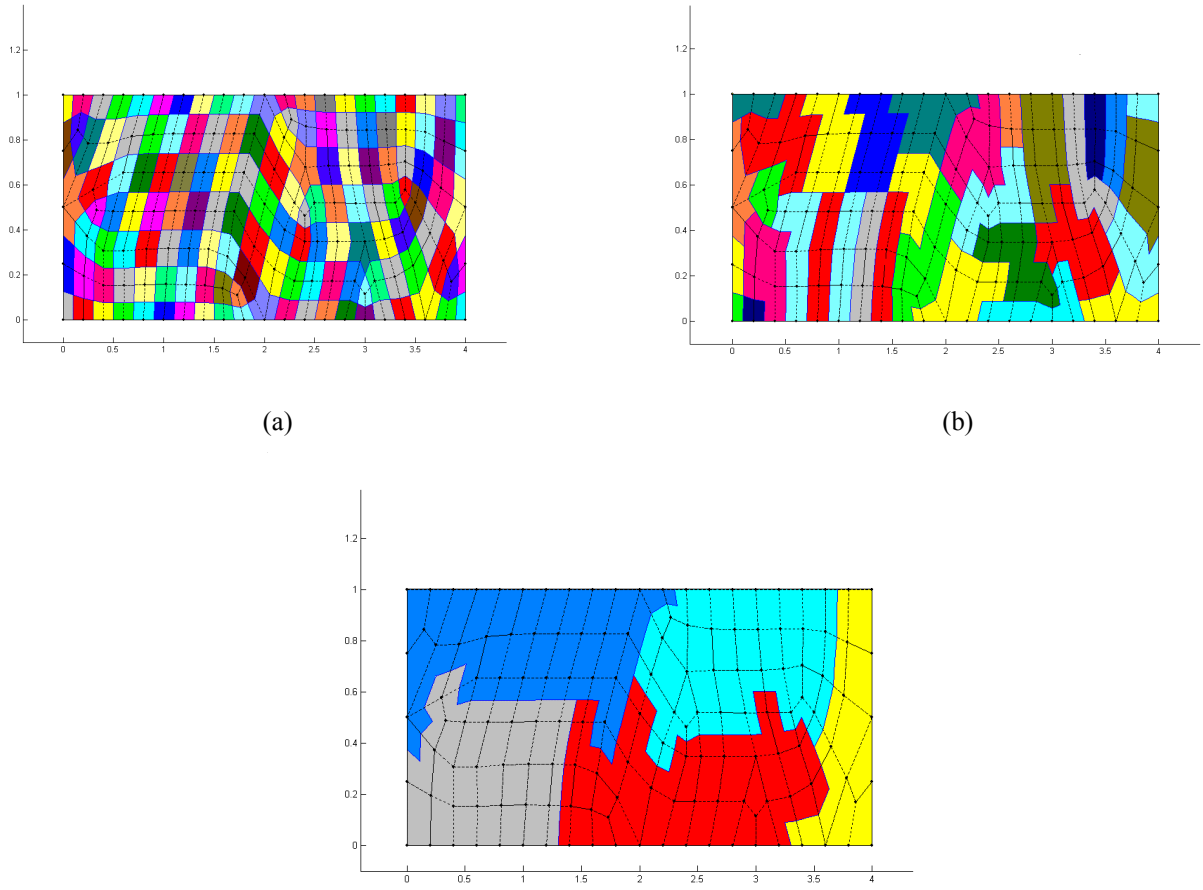
2. Additive Correction Multigrid

The Additive Correction Multigrid method (ACM) distinguishes from conventional multigrid methods for generating the coarse grid equations without the use of fixed stencils. Classical multigrid methods forms the coarse-grid equations by discretizing the governing equations on each grid and interpoling the fine-grid residuals to coarser-grid equations Instead, ACM forms the coarse-grid equations by asserting integral conservation over blocks of control volumes. It determines a constant correction in each coarse grid cell by forcing the sum of residuals be zero after the correction is applied.

In the ACM method the discretization is made only in the finest grid, which makes the computational complexity and cost smaller than other multigrid methods and eliminates the possibility of inconsistent approximations between grids.

The coarse-grid equations are formed by the adding the fine-grid equation requiring that the integral form of the conservation equation be satisfied over the block. Solutions of equations in the coarse grids and its subsequent adjustment to fine grid produce a solution that is conservative on each block. This is a physically desirable property. The study of iterative methods for the solution of discrete linear equations shows that the performance of these methods declines when the coefficients of a linear equation become highly anisotropic within a nodal equation or discontinuous between two adjacent nodal equations. This happens due to the grids aspect ratio, differences between the natural timescale of the transport processes and the time step of the solver iteration.

To exemplify the ACM, an unstructured mesh is depicted in Fig. (1). Although the blocks may be formed in any convenient way, in this example the volumes are joined in blocks with five cells. We can see that the new coarse-grid blocks are formed by joining fine-grid control volumes in different directions, resulting in an agglomeration with high unstructured forms.



(a) (b)

(c)

Figure 1. Example of a adaptive agglomeration in unstructured grids, (a) fine grid (original one), (b) first coarse grid, (c) second coarse grid.

In order to better understand the ACM method, we must show the deduction of the coarse-grid equations. The system of equations to be solved can be written in the common form as

$$A_i \phi_i = \sum_{nb} A_{nb} \phi_{nb} + b_i , \tag{1}$$

where A_i are coefficients in the discrete equation for ϕ of the control volume considered, A_{nb} are coefficients in the discrete equation for ϕ of the neighbors control volumes, b_i is the source coefficient and ϕ is the solution. The coefficients of Eq. (1) can be obtained by applying finite volume or finite difference techniques to structured or not structured grids (Maliska, 2004).

The core of the ACM method consists in using of a correction equation that has the role of adding coarse-grid corrections (ϕ^*) to the best estimate of ϕ on the fine grid:

$$\tilde{\phi}_i = \phi_i + \phi^*_{I,i} , \tag{2}$$

where $\tilde{\phi}_i$ is the improved solution and $\phi^*_{I,i}$ is the correction related to all volumes that lie within the I block.

As it is desired that the corrected solution should have residue sum equal to zero:

$$\sum_i \tilde{r}_i = 0 , \tag{3}$$

thus

$$\tilde{r}_i = -A_i \tilde{\phi}_i + \sum_{nb,I} A_{nb} \tilde{\phi}_{nb} + b_i . \tag{4}$$

Substituting Eq. (2) in Eq. (4) yields:

$$\tilde{r}_i = -A_i \left(\phi_i - \phi^*_{I,i} \right) + \sum_{nb,I} A_{nb} \left(\phi_{nb} - \phi^*_{NB,nb} \right) + b_i . \tag{5}$$

Rearranging the residual equation, we find:

$$\tilde{r}_i = b_i - A_i \phi_i + \sum_{nb,I} A_{nb} \phi_{nb} - A_i \phi^*_{I,i} + \sum_{nb,I} A_{nb} \phi^*_{NB,nb} \tag{6}$$

and it follows that

$$r_i = b_i - A_i \phi_i + \sum_{nb,I} A_{nb} \phi_{nb} , \tag{7}$$

hence, Eq. (4) can be rewritten as

$$\tilde{r}_i = r_i - A_i \phi^*_{I,i} + \sum_{nb,I} A_{nb} \phi^*_{NB,nb} . \tag{8}$$

Replacing Eq. (8) in Eq. (3), we obtain the new equation for the I-block,

$$0 = \sum_{i,I} r_i - \sum_{i,I} A_i \phi^*_{I,i} + \sum_{i,I} \left(\sum_{nb} A_{nb} \phi^*_{NB,nb} \right) . \tag{9}$$

Therefore Eq. (5) can be rewritten as:

$$\sum_{i,I} A_i \phi^*_{I,i} = \sum_{i,I} \left(\sum_{nb} A_{nb} \phi^*_{NB,nb} \right) + \sum_{i,I} r_i \tag{10}$$

or

$$A_p^* \phi_p^* = \sum_{nb} A_{NB}^* \phi_{NB}^* + b_p^*, \tag{11}$$

being this the equation which needs to be solved in order to obtain the correction $\phi_{i,I}^*$. This correction is added to each ϕ value of control-volumes that lie within the I-block. Thus, the improved estimate $\tilde{\phi}_i$ is obtained.

Using the ACM method, there are no further decisions to be made related to the boundary conditions, transfer of residuals or interpolation of variables. These choices are constrained, because ACM requires that the integral conservation over each coarse grid control volume be satisfied

In the next section it is presented the adaptive agglomeration scheme, which improves the performance of ACM method.

2.1. Adaptive Agglomeration Scheme

The nature of the coefficients in the fine mesh equations can strongly affect the convergence behavior of an iterative solver. Therefore, an adaptive agglomeration method based on fine mesh coefficients should be used. This kind of agglomeration becomes more efficient summing up fine grid cells with small timescale of the transport process (large coefficients) and thus obtains a cell with small variation in the timescale.

As it is shown in Elias (1993), the natural timescale $\tau_{i,j}$ for a perturbation in ϕ to travel from the neighbor node i to the cell node j is inversely proportional to the influence coefficient between i and j:

$$\tau_{i,j} = \frac{\rho_{i,j} \Delta s \Delta n}{a_{i,j}}, \tag{12}$$

where Δn is the distance between nodes, Δs is the width of the faces between the nodes, $a_{i,j}$ is the coefficient that represents the influence of a neighbor node j on the cell node i, and $\rho_{i,j}$ is the density at the face. The $a_{i,j}$ coefficient is defined as:

$$a_{i,j} = \frac{\Gamma_{i,j} \Delta s}{\Delta n} + \max\left(-\rho_{i,j} u_m \Delta s, 0\right), \tag{13}$$

where u_m is the face normal velocity and $\Gamma_{i,j}$ is the diffusivity at the face.

An iterative solver applied to a set of discrete linear equations approaches a converged solution by propagating disturbances through the solution domain in an attempt to reduce the residue of the equation set. For iterative methods like Gauss-Seidel and others, there is a timestep over which the solution at node i is advanced in one iteration, namely $\Delta t_i = \min(\tau_{i,NB})$. Thus, the performance of iterative solver degrades when there is a high difference between the implied solver timestep and the timescales to propagate the information between nodes. In order to accelerate the convergence of the discrete linear equations set solution, we can generate a coarse grid equations set by adding cells to reduce the differences between the largest and smallest transport time scales. This is the principle of adaptive agglomeration. (Elias, 1993).

The adaptive agglomeration scheme begins with a single fine grid cell, and through a set of rules it determines which of the neighbors should be included in the coarse grid cell. The new members of the coarse cell are examined to determine which of their neighbors should be included and this is done until the coarse cell reach the desired size, or there are no more neighbors to be added. Then, another coarser grid can be built agglomerating the new cells in the same way, and so on.

Following the nomenclature of a typical family tree as proposed by Elias et al (1997), we call the fine grid cell, whose neighbors are being examined, as the parent, and its neighbors included in the same coarse cell as its children. The current parent's parent is known as the grandparent. This nomenclature is very convenient because the growth of a coarse cell occurs in a tree-like fashion.

There are two main rules used to decide which fine grid cells are added to the new coarse grid cells considered (Elias, 1993). The first agglomeration rule states that a neighbor of a parent cell can be a child if the transport timescale between the parent and the neighbor is of the same order or smaller than the timescale between the parent and the grandparent. Or in other words, the coefficient that connects the parent with the neighbor must be of the same order or larger than the coefficient that connects the parent with the grandparent. Representing the parent by the index i, the neighbor by j and the grandparent by h, we have that j is a possible child of i if:

$$\max\left(a_{i,j}, a_{j,i}\right) \geq \max\left(a_{i,h}, a_{h,i}\right) / 2. \tag{14}$$

The second rule states that a cell is excluded if the interface timescale is very large. The concept of what would be "large" is defined relative to all other timescales which affect the cell in question. A possible child can be agglomerated with the parent if the timescale between the parent and the possible child is of the same order or smaller than the timescale between the child and its other neighbors. In other words, the coefficient that connects the parent with the child must be of the same order or larger than the coefficient that connects the possible child with its neighbors. Thus, taking the coefficients again and representing the parent by the index i, the possible child by the index j and the possible child's neighbors by the index h, j is agglomerated with i if:

$$\max\left(a_{i,j}, a_{j,i}\right) \geq \max\left(a_{j,h}, a_{h,j}\right)/2 . \tag{15}$$

In order to improve the efficiency of this agglomeration scheme some details must be also considered. For instance, to simplify the choice of the first parents for coarse grid cells and ensure that all cells are agglomerated in an only sweep of the fine grid, the first parents are selected in the same order of sweep of the iterative solver. Moreover, as the initial parent doesn't have any grandparent and any preferential timescale, the rule one can be ignored.

Due to the need to eliminate the small timescales, the use of agglomeration rules can lead to the formation of highly unstructured meshes and with large number of coefficients in the coarse mesh equations. To minimize the connections in the coarse mesh, two passes are made through the agglomeration rules on each new level. In the first pass, only the fine mesh cells which already have neighbors (besides the parent) in the coarse mesh cell are examined.

It is possible that there are remaining cells in the fine grid without any neighbor that can be used as child. To eliminate wasted computational effort with those single cells, any parent cells which do not have children are forced to join the coarse grid cell of its most strongly connected neighbor.

In the next section, it is shown the method used to discretizing the differential equation in most problems analyzed in this paper.

3. Element-based Finite Volume Method (EbFVM)

In a finite volume methodology the computational domain is covered by non-overlaping control-volumes where the balances are made. In the EbFVM the element concept and their geometric representation inherited from the finite element method is also utilized, and its definition precedes the creation of the control volume. Figure (2) shows an example of an EbFVM grid, emphazing the differences between grid elements (quadrilaterals) and control-volumes (built around the grid nodes).

The control volumes are created by the method of the medians, which consists of joining the center of the elements to their medians. The resulting control volume is formed by portions (sub-control volumes - scv) of neighboring elements. The integration of the conservation equation over the control volume interfaces is evaluated at the integration points, like the ones shown in Fig. (2).

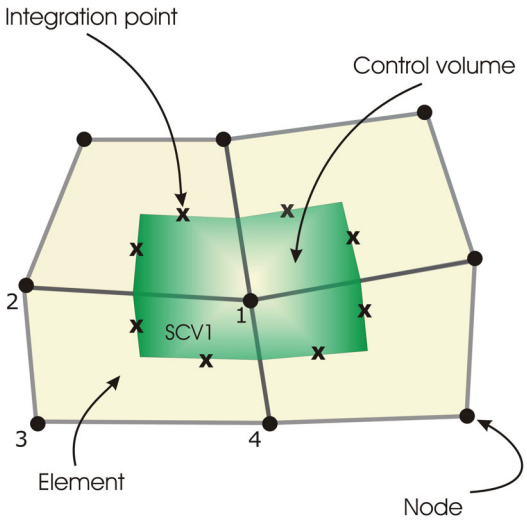
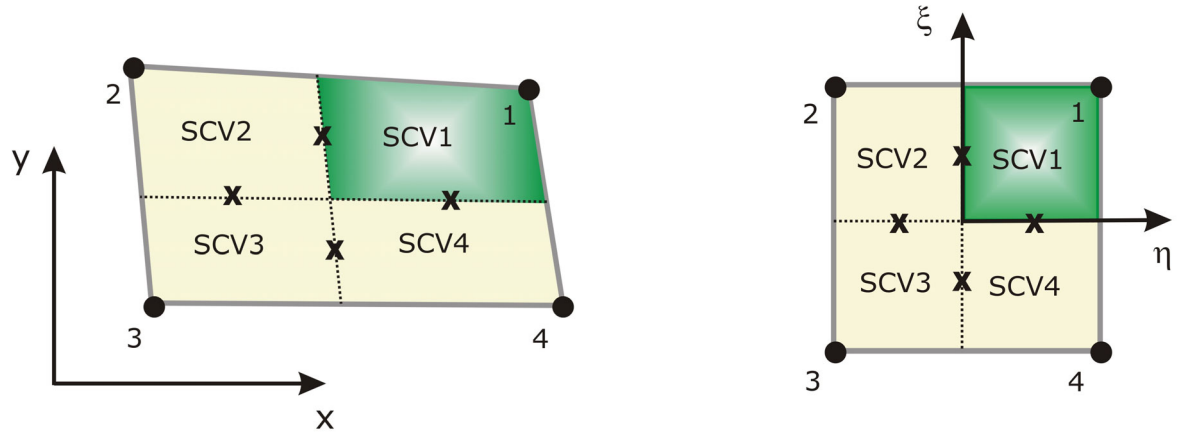


Figure 2. Example of a grid utilized by EbFVM.

Each element in a grid is treated identically in the EbFVM, no matter how distorted the element may actually be in terms of global coordinates. It is possible due to the use of the local coordinates. In Fig. (3) we can see an example of a transformation of an element from the physical plane (x,y) to the transformed plane (ξ, η), whose axes vary $-1 \leq \xi \leq 1$ and $-1 \leq \eta \leq 1$.



(a) (b)

Figure 3. Physical (a) and transformed (b) domains.

The EbFVM will be used to solve simple heat transfer problems, with the main goal of to pointing out the potentialities of the algebraic multigrid. The procedure to obtain the discretized equations will be considered now. The steady-state heat conduction equation, without heat generation is given as:

$$\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right)=0\,, \tag{16}$$

where k is a thermal conductivity, x_j are the global coordinates and T is the temperature.

Integrating the equation for an arbitrary control volume, it results in:

$$\sum_{ip}\left(k\frac{\partial T}{\partial \vec{n}}\right)_{ip}\Delta s_{ip}=0\,, \tag{17}$$

where ip are the integration points, $\vec{n}$ is the normal vector and Δs is the face area.

The shape functions, which are used to describe the variation of the global coordinates within the element, can be used also to describe the variations of T within the element. Thus:

$$T(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)T_i\,, \tag{18}$$

where the shape functions N_i are defined by:

$$N_1(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)\,, \tag{20.1}$$

$$N_2(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)\,, \tag{20.2}$$

$$N_3(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)\,, \tag{20.3}$$

$$N_4(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)\,, \tag{20.4}$$

and the nodal i coordinates x_i and y_i can be write by:

$$x(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)x_i\,, \tag{21.1}$$

$$y(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)y_i\,. \tag{21.2}$$

Therefore, the derivatives of T in Eq. (17) can be determined as

$$\left.\frac{\partial T}{\partial x}\right|_{\xi,\eta} = \sum_{i=1}^{4} \left.\frac{\partial N_i}{\partial x}\right|_{\xi,\eta} T_i \tag{19.1}$$

$$\left.\frac{\partial T}{\partial y}\right|_{\xi,\eta} = \sum_{i=1}^{4} \left.\frac{\partial N_i}{\partial y}\right|_{\xi,\eta} T_i \tag{19.2}$$

where the N derivatives is given by

$$\frac{\partial N_i}{\partial x} = J\left[\frac{\partial N_i}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial N_i}{\partial \xi}\frac{\partial x}{\partial \eta}\right], \tag{22.2}$$

$$\frac{\partial N_i}{\partial y} = J\left[\frac{\partial N_i}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial N_i}{\partial \eta}\frac{\partial y}{\partial \xi}\right], \tag{22.1}$$

where the Jacobian is given by

$$J = \left[\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right]^{-1}, \tag{23}$$

and the derivatives of the shape functions related to ξ and η, in turn, are:

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}, \tag{24.1}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}. \tag{24.2}$$

More details involving the deduction of EbFVM discretized equations can be found elsewhere (Raw, 1985, Maliska, 2002).

4. Results

Three example problems will be presented to demonstrate the efficiency of the ACM method with the agglomeration scheme. The results are obtained using two different solvers: Gauss-Seidel method and ACM method with different number of levels of agglomeration. In the last, the coarsest level equations were solved with a direct-LU solver with the other coarse levels being solved using Gauss-Seidel. The maximum allowed residue was 10^{-5}. We use the V cicle multigrid in all cases analyzed here, although there are other types of multigrid cycles point out in the literature (Trottenberg et al., 2001).

The first example is a square 2D heat conduction problem with anisotropic regions. The geometry and the boundary conditions of this problem are presented in Fig. 4. The discretization was made in traditional FVM (Maliska, 2004) with 20x20 regular volumes.

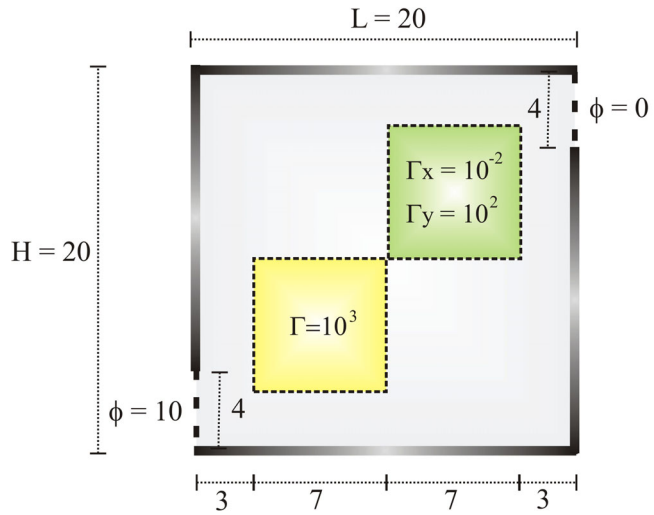
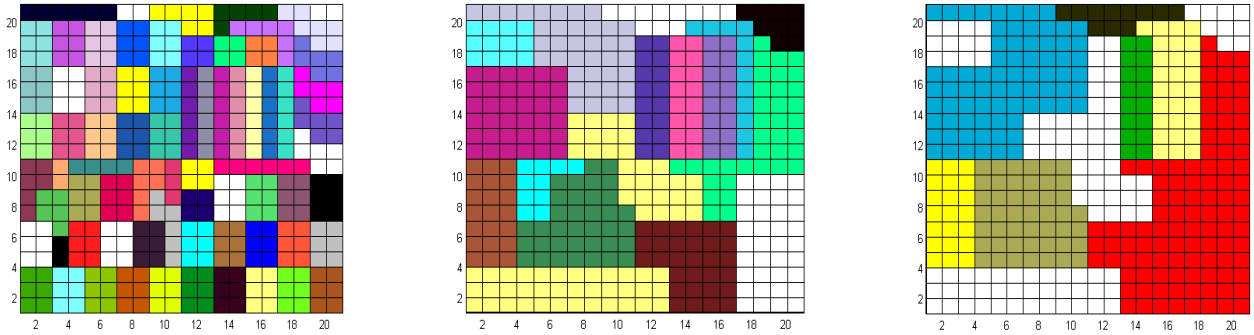


Figure 4. Geometry of diffusion problem example.

The agglomeration of the volumes was performed using 6 fine grid cells per new coarse cell, which result in the meshes shown in Fig. (5).



(a) (b) (c)

Figure 5. Different levels of agglomeration.

We can see that in the first coarse grid, Fig. (5a), the cells generated are reasonably regular. The agglomeration occurs in the direction of a small transport timescales (high diffusivities) in anisotropic zones. In the second and third agglomerations, Fig. (5b) and (5c), coarser blocks continue keeping a good separation of the three diffusivity zones. This behavior was also verified in the work of Elias et al. (1997), and it was expected, since the same basic ideas concerning the agglomeration scheme is used here.

Figure (6) shows the comparison among the convergence behavior on this problem obtained by Gauss-Seidel method and ACM method with different number of agglomeration levels. It is shown that ACM method is more efficient that Gauss-Seidel, as expected. Tests show that this difference in total CPU time tends to grow up when the mesh is more refined. It demonstrates that the adaptive agglomeration, which joins cells with small transport time scales, is a promising technique. In this case, the multigrid with three levels demonstrates to be the best in residual reduction.

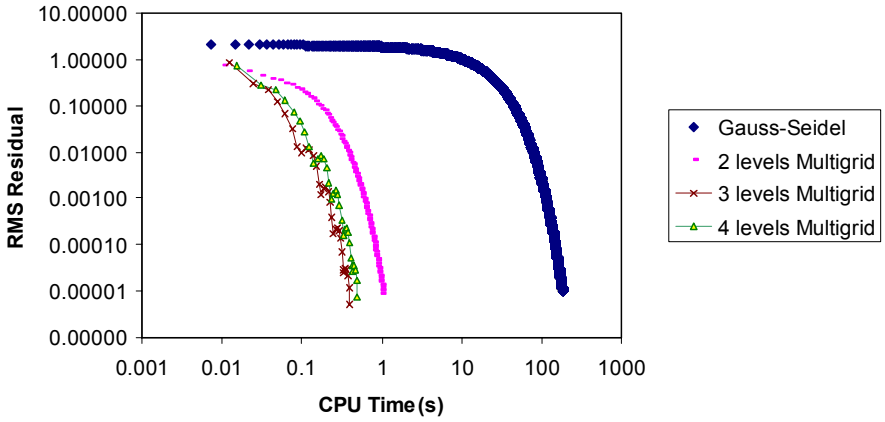


Figure 6. Residual reduction with time.

The second problem considered is a diffusion problem with x-boundaries of the domain insulated and the y-boundaries partly insulated, as represented in Fig. 7. The EbFVM was used for discretizing this problem because it treats the grids in an unstructured fashion, even though it is used here grids geometrically structured with 32x32, 64x64 and 128x128 nodes. Because of the problem geometry, the coefficients in the interior of domain are greater in the y direction than in x direction. The agglomeration is always made in the direction of the greater coefficients.

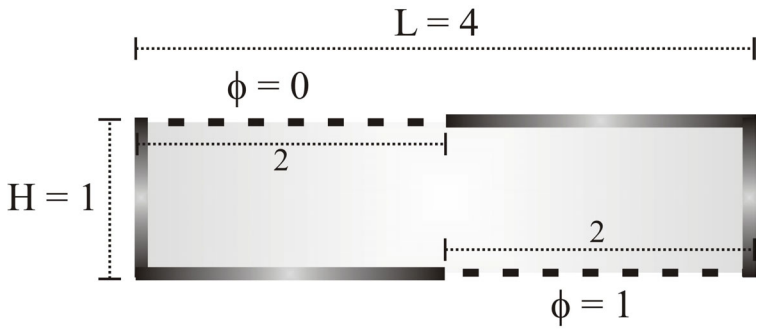


Figure 7. Geometry of the second example problem.

In Fig. (8) the total CPU time needed to reduce the RMS residual to a pre-determined value obtained for different methods is presented. According to that figure, in the 32x32 nodes grid, ACM did not converged to the desired solution faster than Gauss-Seidel. It occurs because the coefficients are large in the y direction and the effect of the Dirichlet boundary conditions is rapidly transmitted to interior of domain. Thus, the residue to be eliminated in the x direction is small and Gauss-Seidel is sufficient to solve this diffusion problem.

However in 64x64 and 128x128 nodes grid, ACM has a significant computational time gain, as shown also in Fig. (8). Due to the increasing of the number of nodes in the y direction, it was expected that the propagation of the Dirichlet conditions has become slower in the y direction when the Gauss-Seidel solver is used. Thus, the ACM is more efficient for using an adaptive agglomeration, which facilitates the information transmission into the interior of domain.

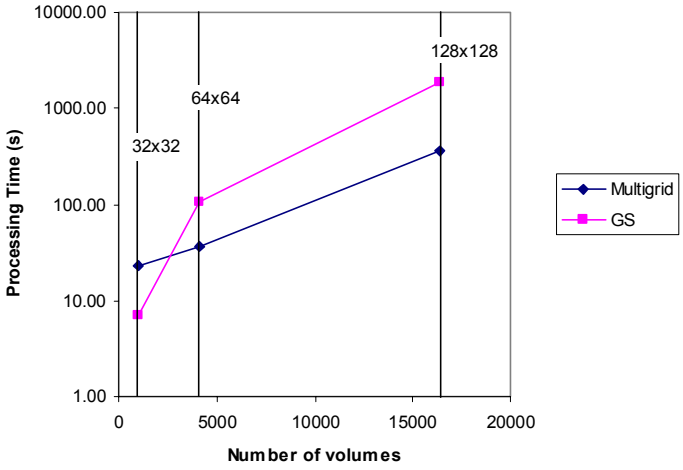


Figure 8. Residual reduction with time in the second problem.

The third example is similar to second one, except that the boundary conditions on x and y coordinates are interchanged (Fig. 9). In this example, the information about the boundary conditions is transmitted mainly in the direction of smaller coefficients, so we expect a larger computational effort required for Gauss-Seidel to reach the convergence.

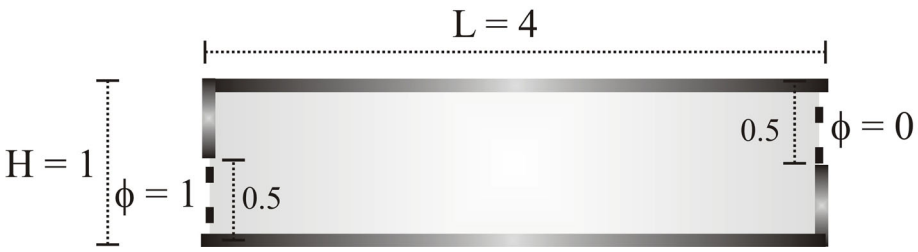


Figure 9. Geometry of the third example of diffusion problem

In Fig. (10), we can see a high computational time gain of ACM over Gauss-Seidel solver. It occurs because the Dirichlet boundary conditions are prescribed in the weak coefficient direction, as already stated. Thus, Gauss-Seidel propagates the boundary condition information in this direction one node per iteration, while ACM propagates the information over the whole domain in only one iteration. As before, Fig. 10 shows that the number of grid nodes grow up the efficiency of ACM in comparison with Gauss-Seidel.

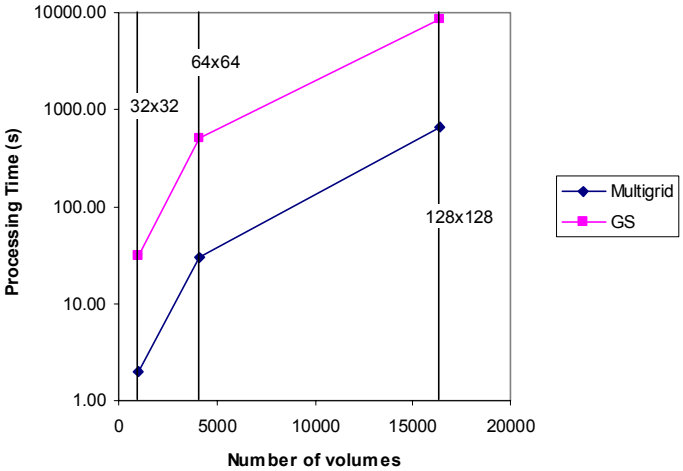


Figure 10. Residual reduction with time in the second problem.

5. Conclusions

This paper has outlined some of the ACM method features. It was shown that the performance of iterative solvers tends to degrade when the matrix coefficients are anisotropic, or the number of equations becomes large. Therefore agglomeration based on geometric factor only are not suitable and the ACM main feature is the adaptive geometric agglomeration, which joins cells to eliminate small transport time scales.

Examples presented in this paper have shown how convergence changes with the degree of anisotropy and with boundary conditions type. The effects of the use of ACM with adaptive agglomeration in these situations were also shown. Moreover, ACM was applied to unstructured meshes, pointing out the flexibility of joining a conservative geometric multigrid method with the EbFVM.

These initial tests are strongly encouraging and efforts are going on foreseeing the application of this principle to a coupled linear equation system, aiming also the development of new features in the adaptive agglomeration scheme, with new rules for varying coarse cells size in each coarse grid.

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