# ANALYSIS OF GRADIENT RECONSTRUCTION METHODS ON POLYGONAL GRIDS APPLIED TO PETROLEUM RESERVOIR SIMULATION

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***Abstract.*** *Polygonal grids are unstructured grids formed by generic control volumes with an arbitrary number of faces. Therefore, they are able to represent complex geometries efficiently. Considering such type of grids, this paper focuses on the analysis of gradient reconstruction methods that can be applied to the discretization of flow models in petroleum reservoir simulation. The purpose of these methods is to approximate the gradient vectors associated with all control volumes of a polygonal grid, employing only discrete values of pressure, related to the control volume centroids. The analyzed methods belong to two main groups: one of them uses the Green-Gauss formula, derived from the divergence theorem, and the other one transforms the gradient reconstruction problem into a least squares problem. In order to determine the gradient reconstruction methods that give more accurate numerical results in a reasonable computational time, several tests are performed. The numerical results are analyzed according to the convergence rate of the pressure gradient and also according to the magnitude of the norm of the truncation error associated with the reconstruction process. The main objective of this work is to determine which reconstruction methods have better cost/benefit ratio. This knowledge could contribute to develop more efficient numerical simulators using polygonal grids.*

***Keywords:*** *polygonal grids, gradient reconstruction methods, finite volume method*

## 1. INTRODUCTION

The main feature of polygonal grids is that control volumes have an arbitrary number of faces. They are constructed from triangular or quadrangular grids to reduce the number of unknowns if these original grids were used as control volumes. In petroleum reservoir simulation, the flexibility of these kind of unstructured grids allows a more accurate geometry representation of the reservoir. However, in order to be able to employ these grids is necessary to develop discretization methods to handle such generic control volumes. A possible alternative is one based on explicit gradient reconstruction. Following that approach, the flow across a control volume face is approximated employing pressure gradients determined from the so-called reconstruction methods.

The main task of gradient reconstruction methods is to approximate gradient vectors, associated with all the control volumes, employing a reduced set of discrete values of a scalar variable, the pressure in the present case. Since a cell-centered scheme is considered here, the discrete pressure values must be associated with the cell centroids. In order to apply a gradient reconstruction method, those pressure values must be known. Figure 1 shows a schematic representation of the role of a gradient reconstruction method. In this paper, three gradient reconstruction methods are described and their performance are evaluated in order to identify those with better cost/benefit ratio.

## 2. GRADIENT RECONSTRUCTION METHODS

The gradient reconstruction methods considered herein can be divided into two groups: the first one is based on the Green-Gauss formula, derived from the divergence theorem, and the second one reduces the gradient reconstruction

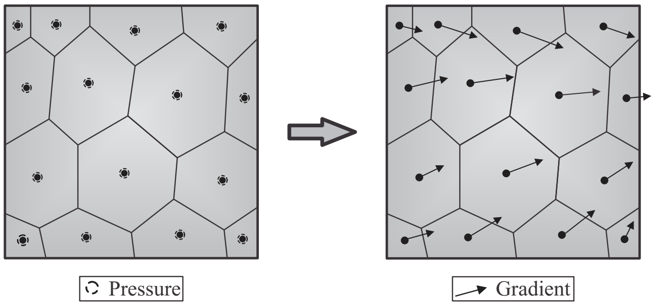



Figure 1. Schematic illustration of the application of a gradient reconstruction method.

problem to a least squares problem.

With the Green-Gauss formula is possible to approximate the gradient vector in an arbitrary control volume through

$$(\boldsymbol{\nabla} P)_p \approx \frac{1}{\Delta V_p} \sum_{f \in p} P_f \, \boldsymbol{\Delta S}_f, \tag{1}$$

where $\Delta V_p$ is the volume of the control volume, $P_f$ is the pressure value at each face centroid on the control volume boundary and $\boldsymbol{\Delta S}_f$ is the area vector associated to each of these faces. As a cell-centered scheme is considered in this work, in order to obtain pressure values at face centroids is essential to use some kind of interpolation. Two interpolation strategies are reported in the literature, the first one is called cell-based approach and the other one vertex-based approach.

The first reconstruction method considered employs the Green-Gauss formula in association with the cell-based approach. This approach utilizes known pressure values, associated to the control volumes that share a given face, to determine the pressure value at this face centroid. The interpolation expression that characterizes the approach can be written as

$$P_f = (1 - \beta_f)\, P_p + \beta_f P_n, \tag{2}$$

where $P_f$, $P_p$ and $P_n$ are the pressure values located at the face centroid and at the control volumes $p$ and $n$, respectively. The weighting factor $\beta_f$ can be calculated considering the following distance weighted average

$$\beta_f = \frac{\boldsymbol{r}_{p,n} \cdot \boldsymbol{r}_{p,f}}{|\boldsymbol{r}_{p,n}|^2}, \tag{3}$$

where $\boldsymbol{r}_{p,n}$ is the vector connecting the centroids of the control volumes $p$ and $n$ and $\boldsymbol{r}_{p,f}$ is the vector that connects the centroids of the face $f$ and the control volume $p$.

The other method that uses the Green-Gauss formula employs the so-called vertex-based approach. Here, face pressure values are determined using vertex pressure values. Since in a cell-center discretization method there are no available vertex discrete values, some averaging procedure must be used, like the one found in Lee *et al.* (2010)

$$P_v = \frac{\displaystyle\sum_{k=1}^{N_v} w_k P_{p_k}}{\displaystyle\sum_{k=1}^{N_v} w_k}, \tag{4}$$

where $N_v$ is the number of control volumes surrounding a vertex of the grid, $P_{p_k}$ represents the pressure values associated with these volumes and $w_k$ represents the weighting factors.

When these weighting factors $w_k$ are determined through the so-called pseudolaplacian procedure, initially proposed by Holmes and Connel (1989), a second order accuracy is ensured for the approximation of the pressure values at the vertices. For this to happen, the condition of null pseudolaplacian must be satisfied,

$$\sum_{k=1}^{N_v} w_k(\boldsymbol{r}_{p_k} - \boldsymbol{r}_v) = 0, \tag{5}$$

where $\boldsymbol{r}_{p_k}$ represents the position vector of a control volume surrounding the analyzed vertice and $\boldsymbol{r}_v$ is the position vector of this vertice.

After obtaining pressure values at all the vertices of the grid using Eq. (4), the pressure values associated to all faces can be determined by a simple arithmetic average. Then, employing the Eq. (1) is possible to finally determine the pressure gradient vector associated with all the control volumes of the grid.

The second group of reconstruction methods employs a strategy that transforms the problem of approximating a gradient into a least squares problem. According to it, the pressure variation in the neighborhood of a control volume is assumed to be linear. Considering variation between the centroids of a control volume $p$ and its neighbor $n_k$, the pressure in the latter one can be written as

$$P_{n_k} \approx P_p + (\boldsymbol{\nabla} P)_p \cdot \boldsymbol{r}_{p,n_k}, \tag{6}$$

where $\boldsymbol{r}_{p,n_k}$ is the vector that connects the centroids of the volumes $p$ and $n_k$, whose Cartesian components are $\Delta x_k$ and $\Delta y_k$. The expression is valid for $k = 1, 2, ..., N_{vc}$, where $N_{vc}$ is the number of neighbor volumes of $p$. The approximation error in the previous expression is second order, because it is actually a truncated Taylor series expansion in which terms of second order and higher are neglected.

Considering all the neighbor control volumes of $p$, it is possible to write Eq. (6) in an alternative manner, thus obtaining a linear system of equations. As is found in Correa *et al.* (2011), this system can be written in the matrix form

$$\begin{pmatrix} \lambda_1 \Delta x_1 & \lambda_1 \Delta y_1 \\ \lambda_2 \Delta x_2 & \lambda_2 \Delta y_2 \\ \vdots & \vdots \\ \lambda_m \Delta x_m & \lambda_m \Delta y_m \end{pmatrix} \begin{pmatrix} \left.\dfrac{\partial P}{\partial x}\right|_p \\ \\ \left.\dfrac{\partial P}{\partial y}\right|_p \end{pmatrix} = \begin{pmatrix} \lambda_1 (P_{n_1} - P_p) \\ \lambda_2 (P_{n_2} - P_p) \\ \vdots \\ \lambda_m (P_{n_m} - P_p) \end{pmatrix}, \tag{7}$$

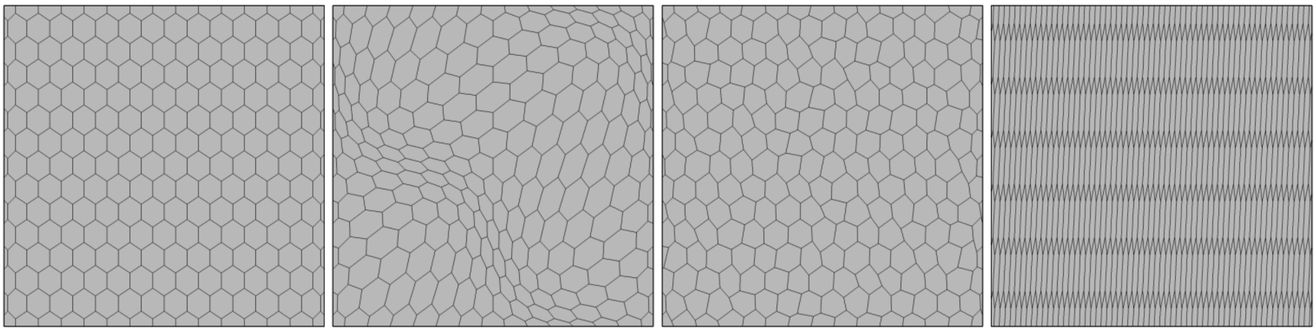
in which $m = N_{vc}$ and $\lambda_k$ are weighting factors. These factors are usually geometric factors that take into account the distance between adjacent control volumes. In the present work an inverse square distance weighting is considered.

It can be noticed in Eq. (7) that the unknowns of the linear system are the two components of the pressure gradient vector, $(\boldsymbol{\nabla} P)_p = (\partial P/\partial x, \partial P/\partial y)_p$, since the pressure values associated with the control volumes are supposed to be known. However, the number of equations, equal to the number of neighboring control volumes of $p$, is usually greater than two. Linear systems like this are called overdetermined and can be solved in the sense of a least squares problem. A lest squares problem can be solved using different techniques (Strang, 1988). That one employed in this work was the QR factorization.

In order to identify the gradient reconstruction methods in a compact way, they will be referred by abbreviations. GGCB will represent the method employing the Green-Gauss formula in association with the cell-based approach, GGVB will represent the one using Green-Gauss with the vertex-based approach, and finally LS will be the one employing the least squares method.

## 3. RESULTS

Tests were performed in order to determine which method has the best performance regarding to the gradient approximation accuracy and the associated computational time. The first aspect was analyzed considering four sets of polygonal grids. Each set is formed by five grids with the same structure and distortion type, but different refinement level. A grid of each set is shown in Fig. 2. The computational time was analyzed considering only one set of grids, called here as regular grids. These grids are formed mainly by hexagons of the same size and shape.



(a) (b) (c) (d)

Figure 2. Examples of grids used in the tests: (a) REG - Regular; (b) SIN - Sinusoidal; (c) RAN - Random; (d) STC - Stretched

The strategy considered was to apply the reconstruction methods to discrete pressure values obtained from analytic functions. Differentiating these functions, it is possible to calculate the exact values of the gradients associated with all the control volumes of the grid. With those exact values, the error associated with each gradient reconstruction method can be measured and its behavior, considering a progressive grid refinement, can also be analyzed. Hence, it is possible to estimate the convergence order related to each reconstruction method. More precise methods show higher convergence orders.

The analysis domain in the tests was a unitary square with the origin located at the bottom left corner. The expressions of the analytic functions, linear and oscillatory, are, respectively

$$P = 5x - 3y, \tag{8}$$

$$P = 5x - 3y + B\left[\mathrm{sen}(7x+1)\,\mathrm{sen}(4y+1)\right]. \tag{9}$$

In order to measure the gradient approximation error on the whole grid, the dimensionless form of the $L_2$ norm of this error was employed (Hurtado, 2011)

$$\varepsilon_{grad} = \left( \frac{\sum_{i=1}^{N_c} |\boldsymbol{\nabla} P_i^{num} - \boldsymbol{\nabla} P_i^{a}|^2 \Delta \mathrm{V}_i}{\sum_{i=1}^{N_c} |\boldsymbol{\nabla} P_i^{a}|^2 \Delta \mathrm{V}_i} \right)^{1/2} \tag{10}$$

where $N_c$ is the number of control volumes of the grid and $\Delta\mathrm{V}_i$ is the volume of each of these entities. On a specific control volume, the numerical gradient and the analytic gradient are represented by $\boldsymbol{\nabla} P_i^{num}$ and $\boldsymbol{\nabla} P_i^{a}$, respectively.

Finally, to estimate the convergence error associated to each gradient reconstruction method is necessary to define a characteristic length of the grid

$$h = \frac{\sum_{i=1}^{N_c} (\Delta\mathrm{V}_i)^{1/2}}{N_c}, \tag{11}$$

where the sum is done considering all the control volumes of the grid.

The numerical results obtained with the reconstruction methods are represented in graphs $\epsilon_{grad}$ x $h$. In the graphs are also included fitted lines, whose inclination is directly related to the convergence order of each method. The dashed lines are only reference lines included for comparison purposes, indicating first and second order error decreasing, respectively.

It was observed in the tests using a linear function that the LS and GGVB methods provide exact gradient values in the entire domain. An explanation for these results can be found in (Cerbato, 2012). In relation to GGCB method, there exists an error even with linear fields, with very low convergence order.

The plot in Fig. 3 (a) shows the reduction of the error norm considering the pressure discrete values coming from the oscillatory function. Since that function is no longer linear, the GGVB and LS methods do not give exact gradients anymore. The estimated value of the convergence order for these two methods are quite close and near second-order. For the GGCB method were obtained the highest values of the error norm and the smallest value of the convergence order, which is close to one.

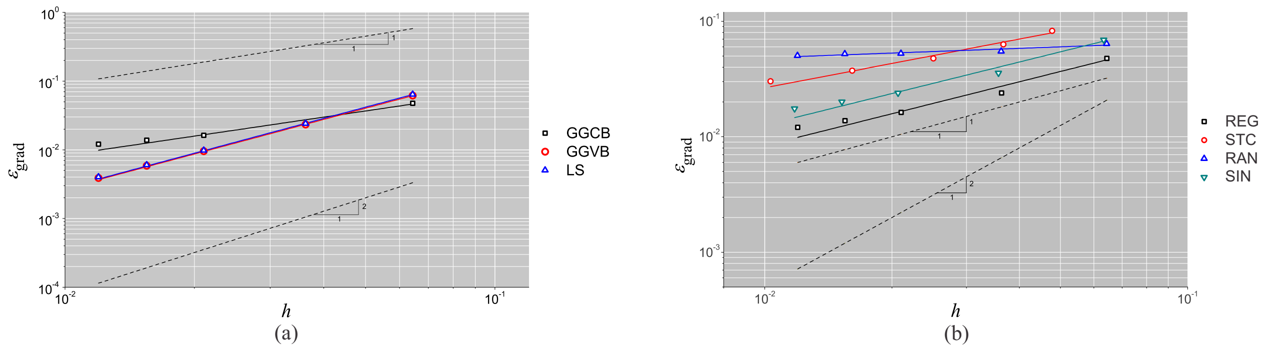



Figure 3. (a) Convergence of the pressure gradient considering the regular grids. (b) Convergence of the pressure gradient in different grids considering the GGCB method.

The purpose of the following tests is to check the sensitivity of the methods with respect to the distortion of the polygonal grids shown in Fig. 2. For simplicity, these grids will be named as: REG (regular), SIN (sinusoidal), RAN (random), and STC (stretched). The tests were performed considering the pressure field obtained from the oscillatory function.

The graph in Fig. 3 (b) displays the behavior of the norm of gradient error employing the GGCB method. It is possible to note that the method fails to adequately treat the grids with random distortion (RAN). The reduction of the error norm with the refinement of the grids is quite low, being the trend line a nearly horizontal line. For the other grids, it is noticed that the trend lines present an inclination, however, it does not even close to first order. As it will be seen in the next graphs, the values of the error norm obtained through GGCB method are the highest among the analyzed methods.

The convergence of the pressure gradient obtained with the GGVB method is shown in Fig. 4 (a). The performance of this method is more uniform varying the types of polygonal grids. The estimated values of the convergence order are higher than unity for all the cases. The difficulty found in GGCB method to deal with the RAN grids no longer exists. Thus, the GGVB method is less sensitive to distortions of the grids than the other method that employs the Green-Gauss formula.

The best performance among the mentioned methods is noticed in Fig. 4 (b), which corresponds to the LS method. This method is insensitive to the distortion of the polygonal grids. As can be seen in the figure, the inclination of the lines are practically the same, only varying the level of the error norm. The behavior noticed in the graph was expected, since the LS method does not depend on the geometry of the control volumes. The method only depends on the positions of the control volume centroids.

The computational time associated with the reconstruction methods was also analyzed. The comparison is done through graphs time versus characteristic length of the grid. Once again the regular grids and the oscillatory function are employed. In order to characterize the computational time, two new concepts were used: the startup time and the reconstruction time. The first one is the time spent with the calculation of fixed parameters, dependent on geometry factors. The weighting factors $\beta_f$ and $\lambda_k$, found in Eqs. (2) and (7), are examples of fixed parameters. On the other hand, the reconstruction time is that one spent on gradient reconstruction operations. These operations employ the parameters calculated at the startup and the current values of pressure. For instance, the reconstruction time associated with the GGVB

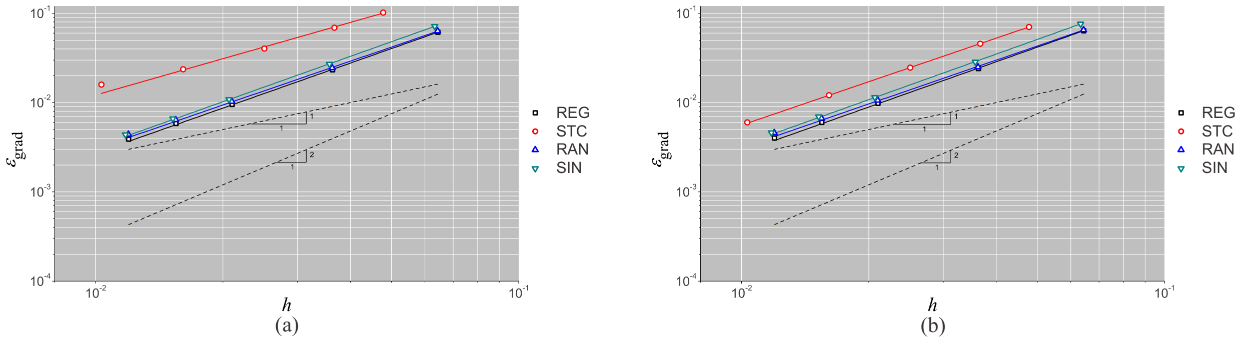



Figure 4. (a) Convergence of the pressure gradient in the different grids considering the GGVB method; (b) Convergence of the pressure gradient in the different grids considering the LS method.

method considers the calculation of the pressure values at the vertices, at the face centroids and also the substitution of those values into the Green-Gauss formula to finally obtain the pressure gradient.

In Figs. 5 (a) and 5 (b) are shown the graphs of the startup and reconstruction times related to the methods GGCB, GGVB and LS. Comparing the reconstruction times, there is no significant difference between the three methods. This is a positive point because it means that the more accurate and robust methods, GGVB and LS, do not consume much more time than GGCB to reconstruct gradients. The higher computational cost of the GGVB and LS methods is on the startup phase, as can be seen in Fig. 5 (a). The parameters calculated at this phase are fixed, thus, it will be executed only once and it will not lead to a substantial difference in the computational times when the reconstruction methods are applied to the solution of differential equations.

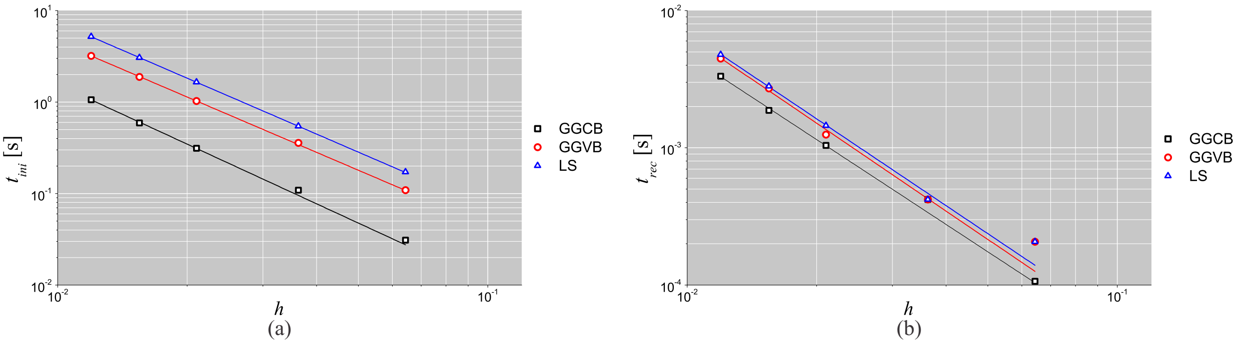



Figure 5. (a) Startup CPU time. (b) Reconstruction CPU time.

## 4. CONCLUSION

Some gradient reconstruction methods applied to polygonal grids were analyzed in the present work. The purpose of those methods is to approximate gradient vectors employing discrete values of a scalar variable. Tests were performed in order to determine which methods would be more adequate for a future application to the discretization of a flow model in petroleum reservoirs using polyhedrical and polygonal grids

The reconstruction methods that exhibit a good cost/benefit ratio, i.e., good numerical results in a reasonable computing time were the method based on least squares approximation (LS) and the Green-Gauss method with the vertex-based approach (GGVB). This superiority is confirmed by the fact that they are more accurate, robust and they do not demand much more time than the other reconstruction method to fulfill their task. When applying the reconstruction methods to even more distorted grids, it is expected that the LS method stand up, since, as mentioned, it does not depend on the geometry of the control volume.

## 5. ACKNOWLEDGEMENTS

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## 6. REFERENCES

Cerbato, G., 2012. *Esquemas numéricos para a reconstrução do gradiente em malhas poligonais*. M.Sc. Dissertation, Federal University of Santa Catarina, Florianópolis, SC, Brazil.

Correa, C.D., Hero, R. and Ma, K.L., 2011. "A comparison of gradient estimation methods for volume rendering on unstructured meshes". *IEEE Transactions on Visualization and Computer Graphics*, Vol. 17, No. 3, pp. 305–319.

Holmes, D.G. and Connel, S.D., 1989. "Solution of the 2d Navier-Stokes equations on unstructured adaptive grids". *Proceedings of the AIAA 9th Computational Fluid Dynamics Conference*, pp. 25–39.

Hurtado, F.S.V., 2011. *Formulação tridimensional de volumes finitos para solução de reservatórios de petróleo com malhas não-estruturadas híbridas*. Doctoral Thesis, Federal University of Santa Catarina, Florianópolis, SC, Brazil.

Lee, H., Jeong, J.J. and Yoon, H.Y., 2010. "An improved numerical scheme to evaluate the pressure gradient on unstructured meshes for two-phase flow analysis". *International Communications in Heat and Mass Transfer*, Vol. 37, pp. 1273–1279.

Strang, G., 1988. *Linear algebra and its applications*. Harcourt Brace Jovanovich Publishers, San Diego, USA, third edition edition.

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