CALCULATING THREE-DIMENSIONAL FLUID FLOWS USING NONORTHOGONAL GRIDS

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ABSTRACT

This paper describes a new solution technique for the prediction of two-dimensional elliptic and three-dimensional parabolic flows. The method solves the conservation equations in a general curvilinear coordinate system maintaining the Cartesian velocity components as dependent variables. The disadvantage in using the Cartesian velocities as dependent variables in nonorthogonal grids is by-passed by forming finite difference equations for the contravariant components in the new system. The PRIME technique [1,2] (update PRessure Implicitly and Momentum Explicitly) is used to handle the pressure velocity coupling problem in conjunction with a new strategy which employs nonorthogonal grids. The method requires slightly higher computer storage but shows virtually the same computational performance when compared with schemes specially designed for Cartesian meshes [3]. The method is tested by solving two-dimensional elliptic and three-dimensional parabolic flows.

1. INTRODUCTION

The prediction of "incompressible" [4] fluid flows in arbitrary regions poses two major problems to the numerical analyst. They are; the strong coupling between pressure and velocity, for flows in which compressibility effects do not dominate, and the domain discretization. The domain discretization has important consequences related to the complexity and generality of the computer code to be designed. If the grid is obtained using an orthogonal system the advantage is that the conservation equations are written in a simple form. This apparent advantage, however, is lost when one faces the complexity in dealing with irregular elements at the boundaries. To overcome this difficulty it seems to require the governing equations be written in a coordinate system which matches the boundary of the domain, defined here a geometrically natural, or boundary fitted coordinate system [5]. These systems may be either orthogonal or nonorthogonal, thus necessitating a decision to be made as which type to use. Orthogonal discretization is numerically attractive since the solution techniques already developed for Cartesian grids can be applied to any orthogonal system with little extra effort.

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However, the generation of orthogonal meshes is not a simple matter due to the necessity in satisfying the orthogonality constraint. In addition, it may be impossible to find an orthogonal mesh for a complex region if some concentration of the coordinate lines is required. In the other hand, the use of nonorthogonal systems has the disadvantage that the equations of motion are somewhat more complex because the presence of the cross derivative terms. These terms will lead to 9-point finite difference equations (for 2-D problems), opposed to a 5-point equations for orthogonal systems. Also, special care is required in choosing the storage location for the dependent variables in the computational grid. A non suitable choice may cause the solution of the 9-point equation for pressure to require excessive computer effort, or it may not converge at all [6].

In spite of that, if fast automatic methods for generating nonorthogonal grids are available, in conjunction with well designed and simple finite difference models, the use of nonorthogonal meshes may be about an optimum alternative for handling fluid flow problems in arbitrary complex geometries. A method which attempts to fullfil the above requirements is now addressed.

2. COORDINATE SYSTEM GENERATION

Consider a three-dimensional irregular domain defined in the Cartesian system, Figure 1a. One wants to map this domain onto a parallelepiped in the (ξ,η,Γ) transformed system, Figure 1b. To realize this objective the following transformation is used

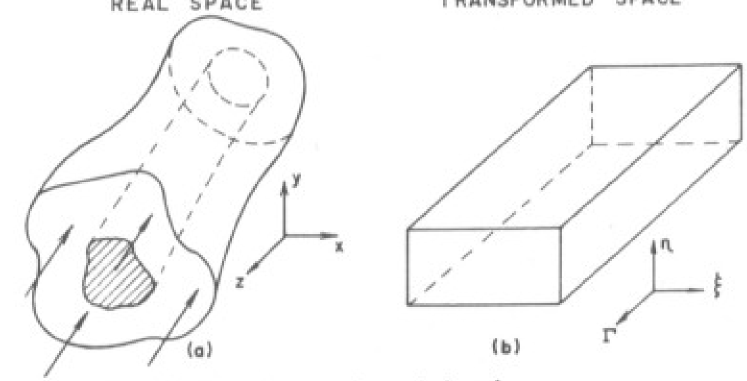

Figure 1. Physical and transformed domains

$$\xi = \xi(x,y,z) \quad ; \quad \eta = \eta(x,y,z) \quad ; \quad \Gamma = z \qquad (1)$$

Following Thompson et al [5], the nonorthogonal coordinate system was generated solving the following linear system of equations

$$\xi_{xx} + \xi_{yy} = P(\xi,\eta) \qquad (2)$$

$$\eta_{xx} + \eta_{yy} = Q(\xi,\eta) \qquad (3)$$

The above system was transformed and numerically solved in the new coordinate system. For details see [5,7].

3. TRANSFORMATION OF THE CONSERVATION EQUATIONS

At this point a decision which has important influences on the overall performance of the model must be made. The question is wheter to have the Cartesian or the contravariant velocity components as dependent variables in the new system. The natural choice would be to use the contravariant components since they are normal to the control volume surfaces and so they carry the mass flow across the element boundaries. This route gives rise to a complex set of transformed equations difficulting the finite differencing process.

If the Cartesian velocities are employed all velocity components need to be calculated at the same point on the grid, so that mass flow across element surfaces can be calculated. The great advantage, however, is that the transformed equations are very simple, making the finite differencing process and easy task. The latter approach is adopted here.

3.1. Transformed Set of Equations

A conservative form of the conservation equations can be written as [8]

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) + \frac{\partial}{\partial z}(\rho w\phi) + P^{\phi} =$$

$$\frac{\partial}{\partial x}(\Gamma^{\phi}\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial y}(\Gamma^{\phi}\frac{\partial\phi}{\partial y}) + \frac{\partial}{\partial z}(\Gamma^{\phi}\frac{\partial\phi}{\partial z}) + S^{\phi} \qquad (4)$$

where ϕ is any scalar field and P^{ϕ} and S^{ϕ} are the pressure and source terms when appropriate. For many duct flow problems encountered in engineering practice the downstream flow conditions have little effect on the upstream flow parameters. The parabolic approximation appears then as a good alternative. In this case, the pressure is split into two terms as follow

$$P(x,y,z) = \tilde{P}(x,y;z) + \bar{P}(z) \qquad (5)$$

Equation (6) is transformed to the new system following the procedure described in [8]. The resulting equation is in the following conservative form

$$\frac{\partial}{J\partial t}(\rho\phi) + \frac{\partial}{\partial\xi}(\rho U\phi) + \frac{\partial}{\partial\eta}(\rho V\phi) + \frac{\partial}{\partial\Gamma}(\rho W\phi) + \hat{P}^{\phi} =$$

$$\frac{\partial}{\partial\xi}(C_1\frac{\partial\phi}{\partial\xi} + C_2\frac{\partial\phi}{\partial\eta}) + \frac{\partial}{\partial\eta}(C_3\frac{\partial\phi}{\partial\eta} + C_4\frac{\partial\phi}{\partial\xi}) + \hat{S}^{\phi} \qquad (6)$$

where U, V and W are the contravariant velocity components written without metric normalization. They are related to the Cartesian velocities by

$$U = y_{\eta}u - x_{\eta}v + (y_{\Gamma}x_{\eta} - x_{\Gamma}y_{\eta})w \qquad (7)$$

$$V = x_{\xi}v - y_{\xi}u + (x_{\Gamma}y_{\xi} - y_{\Gamma}x_{\xi})w \qquad (8)$$

$$W = \frac{w}{J} \qquad (9)$$

where $J = (x_{\xi}y_{\eta} - x_{\eta}y_{\xi})^{-1}$ is the Jacobian of the transformation. The $\hat{P}^{\phi}$ and $\hat{S}^{\phi}$ terms can be found in [7,9]. The coefficients in Equation (6) are given by

$$C_1 = \Gamma^{\phi}J\alpha \quad ; \quad C_2 = C_4 = -\Gamma^{\phi}J\beta \quad ; \quad C_3 = \Gamma^{\phi}J\gamma \qquad (10,11,12)$$

4. FINITE DIFFERENCE PROCEDURE

4.1. Storage Location of the Variables on the Grid

The storage layout where all variables (velocity, pressure, etc.) are stored at the same point, trend followed by most of the recent works dealing with nonorthogonal grids is not adopted in this study. The reason is the danger this approach can pose in having unrealistic pressure and velocity fields [10]. The recommended remedy is the use of a staggered layout [10]. However, the proper staggered layout is not easily recognized when the Cartesian velocities are used as dependent variables in nonorthogonal grids. Then, a storage layout which promotes good convergence characteristics for the Poisson-like equation for pressure should be the main target in design models to predict incompressible fluid flows. Another important feature to be pursued is to have the model reverting to a 5-point scheme when the grid employed is orthogonal. The latter characteristic is numerically attractive since many non-orthogonal grids are, in many situations, quasi-orthogonal or

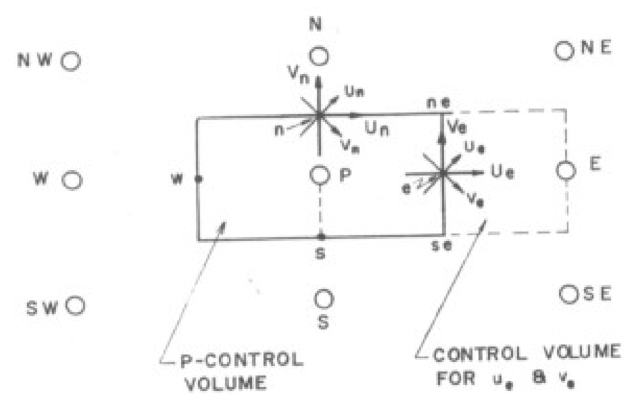

Figure 2. Grid layout

nonorthogonal only in certain regions. The desired flexibility and generality is thus achieved since the scheme is not restricted to the requirements of an orthogonal grid but shares many of its advantages. Figure 2 shows the storage layout adopted in this study. A detailed discussion on this subject can be found in [7,9].

4.2. Discrete Equations

To obtain the discrete equations, Equation (4) is integrated over the elemental control volume shown in Figure 3b. Approximations are introduced [4,11] to reduce the integral equation to the algebraic equation

$$A_p\phi_P^{n+1} = A_e\phi_E^{n+1} + A_w\phi_W^{n+1} + A_n\phi_N^{n+1} + A_s\phi_S^{n+1} + A_u\phi_{P,U}^{n+1} + \frac{A_p}{1+E}\phi_P^n - L\left[\hat{P}^\phi\right]\Delta V + L\left[\widehat{ST}^\phi\right]\Delta V \qquad (13)$$

where

$$A_e = -(\rho U)_e\Delta\eta\Delta\Gamma\left(\frac{1}{2} - \bar{\alpha}_e\right) + \bar{\beta}_e C_{1e}\Delta\eta\frac{\Delta\Gamma}{\Delta\xi} \qquad (14a)$$

$$A_w = +(\rho U)_w\Delta\eta\Delta\Gamma\left(\frac{1}{2} + \bar{\alpha}_w\right) + \bar{\beta}_w C_{1w}\Delta\eta\frac{\Delta\Gamma}{\Delta\xi} \qquad (14b)$$

$$A_p^* = A_e + A_w + A_n + A_s + A_u \qquad (14c)$$

and

$$A_p = A_p^*\frac{(1+E)}{E} \qquad (14d)$$

$$\widehat{ST}^\phi = \hat{S}^\phi + \frac{\partial}{\partial\xi}\left(C_2\frac{\partial\phi}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(C_4\frac{\partial\phi}{\partial\xi}\right) \qquad (14e)$$

In the above equations L[ ] denotes the finite difference approximation of the quantity in brackets and E is a constant that can not exceed unity for explicit formulation.

The discrete equations for the contravariant velocity components are obtained using Equation (13) with ϕ equal to u and v and Equations (7) and (8). For the control volume depicted in Figure 2 the U_e and V_n equations are

$$U_e = \hat{U}_e - \left(\frac{\Delta\Gamma}{A_p^u}\alpha\right)_e(P_E-P_P) + \left(\frac{\Delta\Gamma}{4A_p^u}\beta\right)_e(P_{NE}+P_N-P_{SE}-P_S) \qquad (15)$$

$$V_n = \hat{V}_n - \left(\frac{\Delta\Gamma}{A_p^v}\gamma\right)_n(P_N-P_P) + \left(\frac{\Delta\Gamma}{4A_p^v}\beta\right)_n(P_{NE}+P_E-P_{NW}-P_W) \qquad (16)$$

Similar equations can be written for U_w and V_s, completing then the four velocity components needed for the mass conservation balance applied to the elemental volume shown in Figure 3b. The equation is

$$\left((\rho U)_e-(\rho U)_w\right)\Delta\eta\Delta\Gamma + \left((\rho V)_n-(\rho V)_s\right)\Delta\xi\Delta\Gamma + \left((\rho W)_P-(\rho W)_U\right)\Delta\eta\Delta\xi \qquad (17)$$

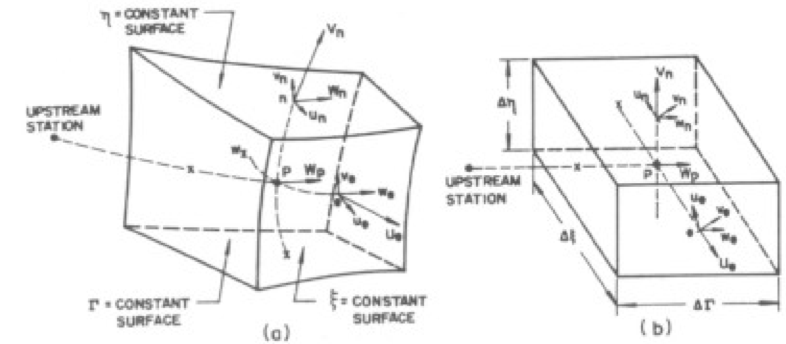

Figure 3. Physical and transformed elemental volumes

4.3. Solution Procedure

In the present work only 3-D flows with a predominant flow direction are considered. Computer storage for the variables are needed only in the calculation and upstream plane. The pressure velocity coupling in the axial direction is solved using the method developed by Raithby and Schneider [4] which removes the necessity of iteration by taking advantage of the linear relationship between the axial velocity and the pressure gradient. The elliptic coupling problem is solved using the PRIME method [1,2]. This method simplifies considerably the iteration cycle since it eliminates the standard two-step solution procedure in which an estimate of the velocity field is obtained by solving the momentum equation with a guessed pressure field, followed by the solution of two Poisson equations for velocity correction and pressure updating [4,10,9].

In the PRIME procedure a Poisson equation is formed by inserting Equations (15) and (16) into continuity equation. The solution of the resulting equation will give a pressure field which is used to correct the hat-velocity vector and at the same time is recognized as the update pressure field. In this technique the momentum equations are solved in a Jacobi iteration fashion. The great attractiveness of this approach is the simplicity it introduces in the computer code. The determination of the hat-velocities uses the best estimates of the velocity field.

The following steps constitutes one iteration cycle performed in the equation set when solving 3-D parabolic fluid flow problems.

(a) Guess the cross pressure field in the inlet section and the pressure gradient $d\bar{P}/dz$. Specify a 3-D velocity profile at the duct inlet.
(b) Compare the coefficients for the axial momentum equation. Solve for w and compute the mass flow. Applying the method described in [4] determine w and $d\bar{P}/dz$. Compute W.

(c) Compute the coefficients for the equations for U and V and compute the hat-velocities. See [7] for details.

(d) Form and solve the Poisson-like equation for pressure. The equation is

$$A_P P_P = A_e P_E + A_n P_N + A_s P_S + A_w P_W + A_{ne} P_{NE} + A_{se} P_{SE} + A_{sw} P_{SW} + A_{nw} P_{NW} + B \quad (18)$$

and

$$A_p = A_e + A_w + A_n + A_s \quad (19)$$

It is seen that the pressure at a point P is strongly linked with the four parallel pressure points and not with the diagonal ones. This is the type of structure exhibited by the Poisson equation derived for orthogonal grids using the standard staggered layout. This similarity is probably the reason why this equation shows similar convergence rate as the equations for orthogonal meshes. In addition, all the diagonal coefficients vanish when the grid employed is orthogonal. The equation was successfully solved using the S.O.R. point iteration method.

(e) Correct the contravariant velocities using Equations (15) and (16). These velocities now satisfy mass conservation.

(f) Determine the contravariant velocities that do not enter the mass conservation balance using an interpolation process.

(g) Determine the Cartesian velocities using Equations (7), (8) and (9).

Iteration, by cycling back to step (b) is required to account for nonlinearities and inter-equation coupling.

5. APPLICATIONS

The assessment of a 3-D parabolic model must begin by testing its secondary flow model. The reason is because the main flow is very little affected by changes in the cross flow and, consequently, the checking of the flow parameters in the predominant direction only does not validate the full model. Furthermore, the experience gained in dealing with the 2-D elliptic flows in nonorthogonal grids is, in principle, extendable to 3-D elliptic problems. These facts require that the model be tested by first solving 2-D elliptic flows such that the cross flow velocities can be compared quantitatively.

The driven flow in a square cavity with a moving lid is a good test problem since flow conditions ranging from dominant diffusion to dominant convection can be analysed. Figure 4a shows the geometric parameters and the boundary conditions for the problem. A nonorthogonal grid with 28 × 28 grid points shown in Figure 4b was used. The problem was also solved in a 28 × 28 Cartesian grid and the solutions compared. Figure 5 shows the u-velocity along the lines A-A and B-B in the nonorthogonal grid for R_e = 400. In the Cartesian grid the lines A-A and B-B

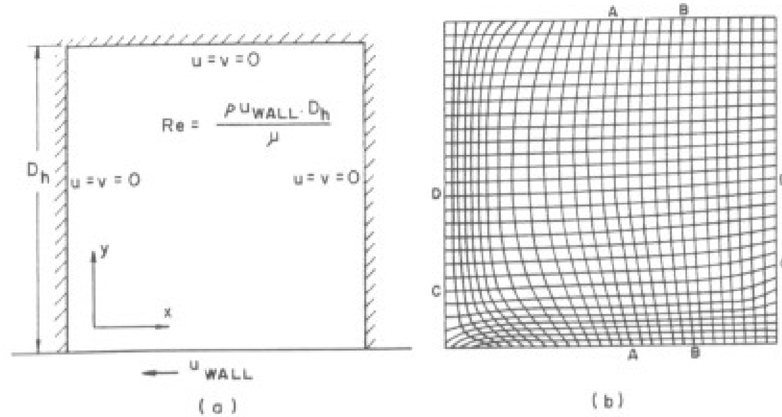

Figure 4. Square cavity problem

are constant x lines with the values 14Δx and 20Δx. The results obtained using orthogonal and nonorthogonal grids are in very good agreement. A slightly disagreement is observed, for a small region close to the maximum reversal velocity, for the results obtained in this study compared to those of Burggraf [12]. This disagreement is noted for both coordinate systems used, what suggests that the problem is not due to the presence of nonorthogonalities. The most probable cause is that the grid was not refined to the point where numerical diffusion no longer influences the solution. Figure 6 shows the v-velocity along the C-C and D-D lines. In the Cartesian grid these are lines of y = 6Δy and y = 14Δy. Again the results using both grids compares very well. Figure 7 brings to the reader the comparisons of the convergence rate obtained using both coordinate systems. It can be seen that the number of iterations needed to obtain a solution of the equation set, to the same level of accuracy, is practically the same for both grids. This finding encourages further developments in this area. As a second test problem the parabolic flow in a converging-diverging duct with changing cross section was solved. The duct geometry

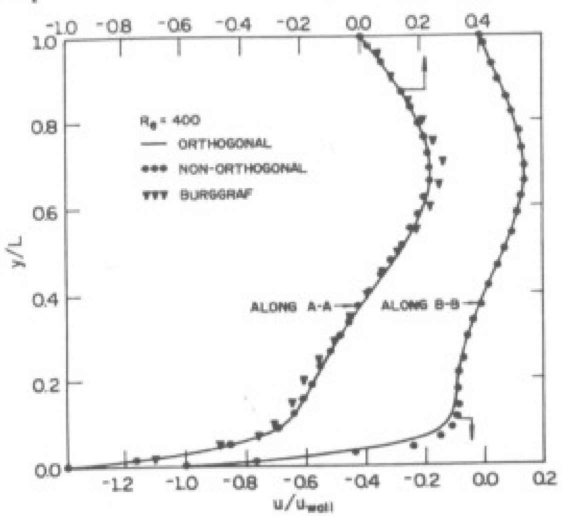

Figure 5. Square cavity, u - velocity

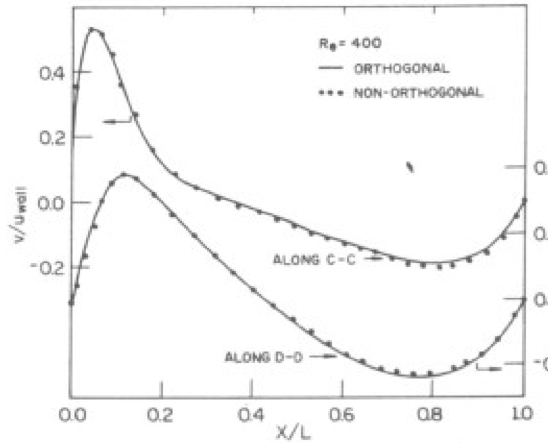

Figure 6. Square cavity, v - velocity

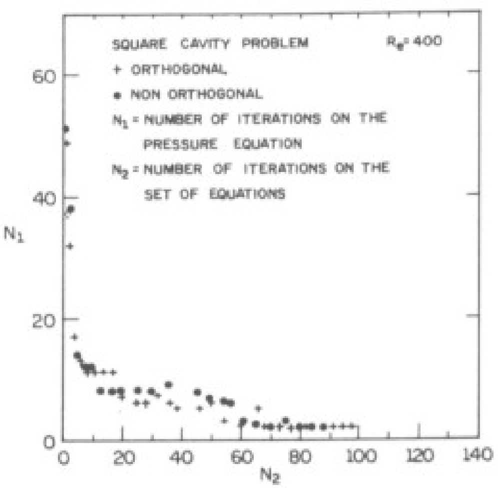

Figure 7. Convergence rate comparison

is shown in Figure 8 and the geometric characteristics of the intermediate calculation planes can be found in [7]. The objective in solving this problem was two-fold. Firstly, due to the duct convergence the flow exhibits strong cross velocities, situation not encountered in the entrance region of constant cross section duct flows. Secondly, the existence of a different nonorthogonal coordinate system for each calculation plane constitutes a good test for the generality of the technique. The axial step was 1/6th of the inlet hydraulic diameter and nine solution planes were used. The velocity profile of the fully developed flow in a rectangular duct was prescribed at the inlet. Figure 9 shows the axial velocity profiles along the symmetry plane x = 0. Comparisons are not made but it can be seen that the axial velocity shows the expected trends. The cross flow velocities, not reported here, also shows the expected trends.

6. CONCLUSIONS

A solution technique for the prediction of three-dimensional parabolic incompressible fluid flow in arbitrary geometry has been presented. The procedure employs non-orthogonal grids and transforms the conservation equations to the new curvilinear system maintaining the Cartesian velocity components as dependent variables.

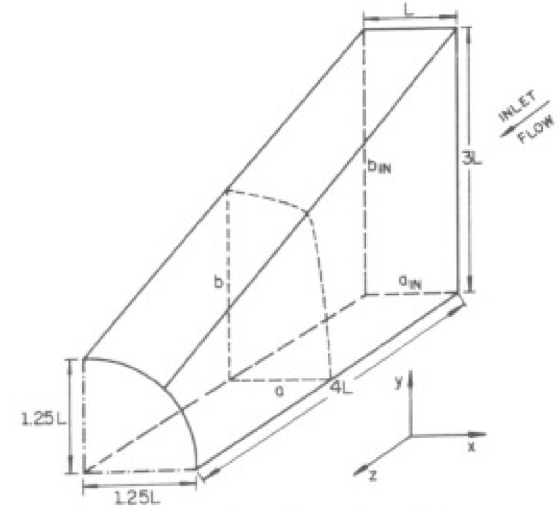

Figure 8. Converging-diverging duct

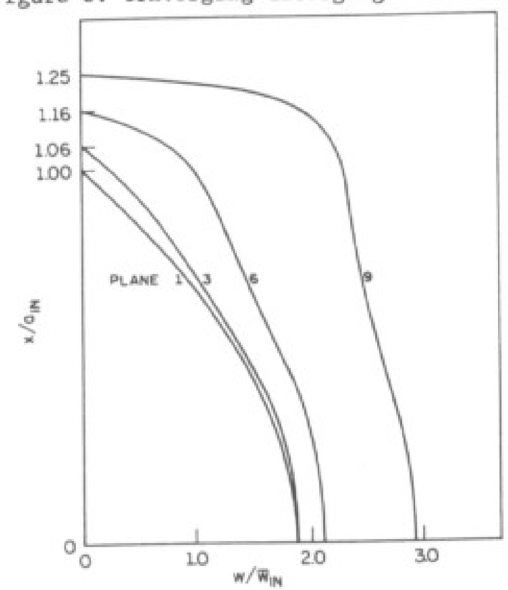

Figure 9. Axial velocity profiles

In the new system finite difference equations for the momentum equations are written in terms of the contravariant velocity components. These equations, togheter with the mass conservation equation, are solved according to the PRIME procedure. The incorporation of the PRIME procedure into a method which employs non-orthogonal grids was of great benefit since it introduced the desirable simplicity to the solution procedure as a whole.

By the tests performed if is seen that the numerical technique presented here is very promising. Currently, tests are being performed where inflow-outflow and thermal problems are being solved.

ACKNOWLEDGEMENTS

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SECTION 9

NON-NEWTONIAN FLOW