A Finite Volume Method for the Solution of Fluid Flows Coupled with the Mechanical Behavior of Compacting Porous Media

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Abstract. A Finite Volume Method is employed for solving the coupled fluid/structure problem encountered in porous deformable media, namely the flow in the porous rock of a petroleum reservoir coupled with the stress field in the porous matrix. As oil is produced, the porous pressure diminishes changing the porosity of the structure, requiring careful monitoring of the structure of the matrix to avoid rock collapse. This coupled problem is normally solved using two different methodologies, finite volume, or finite-differences, for the flow in the porous media, and finite element for the rock mechanics. The reason for this approach is the belief that finite volume methods are suitable only for fluid flow problems. In fact, the momentum conservation equation is, obviously, the same for both problems, allowing the proposition of a single method, using a single grid for solving the coupled problem. As a consequence, it is not required grid interpolation to transfer results from one problem to another and, even more attractive, the approximate equations are conservative for both problems. This paper presents such a solution, whereby the mechanical behavior of the rock is modeled using the Biot's Linear Poroelasticity Theory [1], and the Principle of Effective Stresses [2], while the flow is modeled by the Darcy's equation. The numerical solutions are obtained for 2D domains for a plane strain condition. The method is evaluated by solving the classical problem of porous media compaction with comparison with available analytical solutions. The advantage of having a conservative scheme for the rock mechanics problem, only attainable using a finite-volume method, is clearly demonstrated. The unified numerical approach presented herein is an encouraging alternative for solving coupled problems of engineering interest.

Keywords: coupled problems, finite volume method, porous media flow, compacting porous media
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INTRODUCTION

There exist many engineering problems which involve different physical phenomena, like fluid flow, heat transfer, elasticity, electromagnetism etc, which are required to be solved coupled, since the phenomena influences each other. In this class of problems one encounters the porous media compaction of petroleum reservoirs, which involves the fluid flow and the rock mechanics in the porous media. The mechanical behavior of the rock close to the production wells are of utmost importance, dictating the oil production rate to avoid the fluid porous pressure to reach values below those in which the rock integrity would be at risk. The literature offers several alternatives for solving this problem, with different numerical methods and different algorithms for taking into account the coupling. It is common to employ the Finite Element Method (FEM) for solving the geo-mechanical problem and a Finite Volume Method (FVM) for the flow in the porous media. The reason for this choice are based on the belief that the governing equations of the mechanical problem can be better solving using FEM. However, in the last decades [5,6] it has been shown that mechanical problems can be efficient and accurately solved using FVM, with the advantage of having a conservative scheme, which has been proved to help the robustness of the method. Besides that, conservative schemes guarantee that no spurious source or sinks of the properties are generated, what would be physically inconsistent and deleterious for the whole problem. Using only one numerical method, a single grid is used, avoiding data interpolation, which is a cumbersome routine and may introduce additional errors. Pressure, displacements and porosity are all calculated at the same point in both problems due to the flexibility of using a single grid.

For each time level the equations are indenpendtly solved and iterated until convergence. Pressure and displacements could be also calculated implicitly but, due to the different time scales of the problems, iterating between the two linear systems provides more flexibility in adjusting numerical parameters for convergence. For example, it is possible to delay one solution respect to other, strategy that can't be done in a fully implicit solution. In this work an orthogonal grid is employed, but the methodology can be extended with no difficulties for unstructured grids. The main goal of the work is the use of a FVM for solving the rock mechanics problem, added to the solution of the fluid flow problem in a porous media. Extension of this work can be done for solving the coupling between thermal stresses and temperature and other coupled problems. It is noteworthy to mention that using the conservative form of the momentum conservation equation for the mechanical problem avoids convergence difficulties encountered when non-conservative methods are employed.

PHYSICAL FORMULATION

The governing equations of the fluid flow and the rock mechanics problems are now given.

Rock Mechanics Problem

As mentioned, the governing equations for the stress field in the porous matrix are derived from the Biot's Linear Poroelasticity Theory and from the Effective Stress Principle. The solid matrix is considered isotropic and homogeneous formed by solid compressible grains. Therefore, for a plane strain condition, one has

$$\frac{\partial(\sigma_{xx})}{\partial x}+\frac{\partial(\sigma_{xy})}{\partial y}=0 \qquad (1)$$

$$\frac{\partial(\sigma_{yx})}{\partial x}+\frac{\partial(\sigma_{yy})}{\partial y}=0 \qquad (2)$$

with the stresses given by

$$\sigma_{xx}=\lambda\left[\frac{1-\nu}{\nu}\varepsilon_{xx}+\varepsilon_{yy}\right]-\alpha p \qquad (3)$$

$$\sigma_{yy}=\lambda\left[\varepsilon_{xx}+\frac{1-\nu}{\nu}\varepsilon_{yy}\right]-\alpha p \qquad (4)$$

$$\sigma_{xy}=2G\varepsilon_{xy} \qquad (5)$$

in which, $\sigma_{ij}, p, \lambda, \nu, G, \alpha$ are the components of the stress tensor, fluid pressure, Lamé's constant, Poisson's coefficient, the transversal elasticity module and the Biot's coefficient, respectively. The deformation ε_{ij} is given by

$$\varepsilon_{xx}=\frac{\partial u}{\partial x} \qquad (6)$$

$$\varepsilon_{yy}=\frac{\partial v}{\partial y} \qquad (7)$$

$$\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \qquad (8)$$

where u e v are the components of the displacement vector in the directions x e y, respectively.

Fluid Flow Equations

The governing equations for the single phase porous media flow are the mass conservation equation, given by

$$\frac{\partial(\rho_f\phi)}{\partial t}+\nabla\cdot\left[\rho_f\left(\bar{v}+\phi v_s\right)\right]+q=0 \qquad (9)$$

in which v_s, ρ_f, ϕ, q are the solid velocity, density of the fluid, porosity and the source term, respectively. The Darcy's velocity $\bar{v}$ is related to the pressure gradient by

$$\bar{v}=-\frac{K}{\mu}\cdot\left(\nabla p-\rho_f g\right) \qquad (10)$$

in which K is the absolute permeability tensor, μ is the fluid viscosity and g is the gravity vector. Substituting (9) in (8) and rearranging terms, one gets the final flow equation, as

$$\left[c_f\phi+c_s(\alpha-\phi)\right]\frac{\partial p}{\partial t}-\nabla\cdot\left[\frac{K}{\mu}\cdot\left(\nabla p-\rho_f g\right)\right]+q=-\alpha\frac{\partial\varepsilon}{\partial t} \qquad (11)$$

in which, ε, c_f e c_s are the volumetric deformation, fluid and solid compressibilities, respectively. An equation for updating the porosity [3] is given by

$$\frac{\partial\phi}{\partial t}=(\alpha-\phi)\frac{\partial\varepsilon}{\partial t}+c_s(\alpha-\phi)\frac{\partial p}{\partial t} \qquad (12)$$

Therefore, the above equation set can be solved for the unknowns u, v and p, the primary variables to be solved by the numerical algorithm.

NUMERICAL FORMULATION

The approximate equations are obtained using a Finite Volume Method, that is, by integration of the governing equation in its divergence (conservative) form. The resulting equation is identical as to perform a balance of the property over a discrete control volume [4]. The discretization gives rise to a linear system of algebraic equations for the variables. The variables in the computational grid are located in a staggered fashion, according to Fig. 1.a, with the displacements u e v located at the faces of a pressure control volume. In this arrangement the variables pressure, volumetric deformation and porosity are calculated at the same position in the grid. Detail of this technique can be found in [4]. To demonstrate the integration procedure, the equilibrium equation for the x direction is used. For a control volume p, depicted in Fig. 1.b), one has

$$\int_{w}^{e}\int_{s}^{n}\frac{\partial(\sigma_{xx})}{\partial x}dxdy+\int_{w}^{e}\int_{s}^{n}\frac{\partial(\sigma_{xy})}{\partial y}dxdy=0 \qquad (13)$$

resulting in

$$\left[\sigma_{xx}\big|_{e}-\sigma_{xx}\big|_{w}\right]\Delta y+\left[\sigma_{xy}\big|_{n}-\sigma_{xy}\big|_{w}\right]\Delta x=0 \quad (14)$$

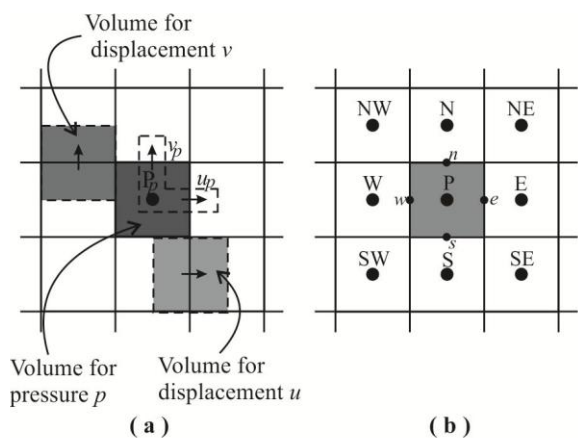


FIGURE 1. (a) Variable arrangement in the grid. (b) Control volume centered at p and its neighbors.

It can be observed in Eq. (13) that the integration of the conservative form of Eqs. (1) and (2) guarantee that the fluxes of momentum, given by the normal and tangential stresses, naturally appear at the control volume interface. Fluxes are the natural boundary condition for the finite volume method and the constitutive equations for the stresses should not be substituted into Eqs.(1) and (2) previously to integration. If this is done there is no conservation of momentum in the control volume, which is equivalent in having a force imbalance for that control volume, what is physically inconsistent. In methods which do not use the concept of control volume for creating the approximate equations this problem is not "seen", but its effects are reflected in the robustness and accuracy of the method. Recall that the goal of the numerical method is to solve the conservation equation at point level (momentum balance at the point). If the solution at a point level is not possible, it seems that the physical consistent approach is to solve its discrete counterpart of the equations, as done by the finite volume techniques [4]. Therefore, Eqs. (6) to (8) should be substituted in Eqs. (3) to (5) and then into Eq. (14).

To close the equation system one needs to add the approximate equations for the boundary control volumes. The boundary conditions can be of two types:

a) *prescribed displacements*. In this case the required stresses (flux of momentum) are calculated using the constitutive equations;

b) *Prescribed stresses*. In this case the values are directly subtituted in Eq. (14), since this is the natural boundary condition for a finite volume method.

The final equation system for calculating the displacements u and v is given by

$$\begin{bmatrix} A^{uu} & A^{uv} \\ A^{vu} & A^{vv} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} B^{u} \\ B^{v} \end{bmatrix} \qquad (15)$$

in which A^{uu}, A^{uv}, A^{vu} and A^{vv} are the [2x2] block matrices of the coefficients and B^{u}, B^{v} are the independent vector containing the fluid pressure from the solution of the porous media flow.

The algebraic equations for the fluid flow problem are obtained in the same fashion, that is, integrating Eq. (11) in space and time. Adopting a fully implicit formulation and approximating the derivatives at the control volume interfaces using central differences, the resulting linear system for pressure is

$$\left[A^{p}\right]\left[p\right]=\left[B^{p}\right] \qquad (16)$$

The coupling term in Eq. (11) involving the deformation of the porous rock is considered known when Eq. (16) is solved, since this term was calculated when the geo-mechanical problem was solved. This term is inside the independent vector of Eq. (16).

The boundary conditions considered for the flow problem are prescribed pressure and zero mass flow (impermeable boundary). The iterative procedure for solving Eqs. (15) and (16) is:

1. Calculate the initial conditions, obtaining the equilibrium between the fluid pressure and the stress in the solid matrix;

2. Advance to the next time level;

3. Estimate the values of ϕ and ε. Normally, they are taken from the previous time level;

4. Calculate the coefficients for A^p and the independent term;

5. Solve Eq. (16) for p.

6. Calculate the coefficients A^{uu}, A^{uv}, A^{vu} and A^{vv} and the independent terms B^u and B^v;

7. Solve the system of equations given by Eq.(15), for u and v;

8. Calculate the volumetric deformation and update the porosity by Eq. (12). Update A^p and ;

9. Iterate from step 4 to 8 up to specified convergence. The convergence is checked using the pressure p;

10. Calculate the stresses using Eqs. (3) to (5);

11. Back to step 2, until the desired simulation time is reached.

NUMERICAL EXAMPLES

Two different problems are used for evaluating the method advanced in this paper. The analytical solution of Problem 1 is used to validate the method, while Problem 2 aims to present a more complex physical situation. Table 1 presents the data for the porous media and for the fluid used in the simulations.

TABLE 1. Data for test cases

	Problem 1.	**Problem 2.**
Material	Berea sandstone	Ohio sandstone
No. of elements	1200	1000
Time step [sec]	1	3600
σ_0 [MPa]	1	10
p_f [MPa]	0	0
G [MPa]	6.0×10^3	6.8×10^3
ν	0.2	0.18
α	0.79	0.74
K [md]	1.9×10^2	5.6
c_s [Pa^{-1}]	2.65×10^{-11}	3.11×10^{-11}
c_f [Pa^{-1}]	Incompressible	3.03×10^{-10}

Problem 1. Poroelastic Column

This problem consists in a porous column containing a incompressible fluid and with the boundary conditions given in Fig. 2. The initial (equilibrium) condition is obtained applying the load σ_0 over the column considering that, in that time no fluid are allowed to cross the boundaries. Therefore, at time equal to zero the applied load is supported by the fluid and by the solid. Just after this time, the fluid is allowed to escape by the top of the column, initiating the transient compaction problem.

Biot[1] solved this problem analytically, obtaining expressions for the transient pressure at the bottom of the column and the displacements in the y direction at the top of the column.

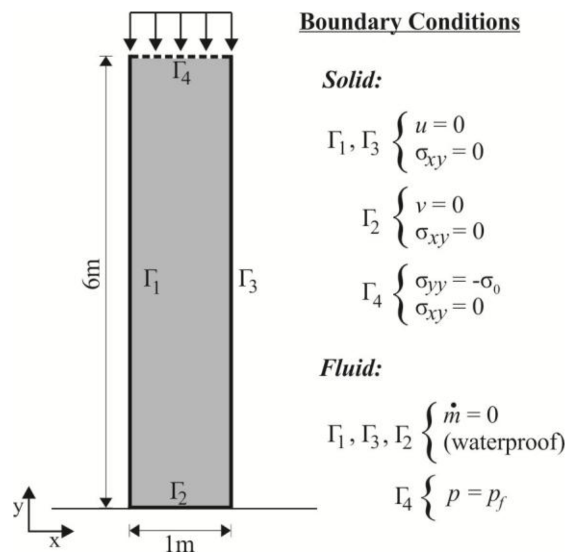


FIGURE 2. Solution domain and boundary conditions for Problem 1.

Comparisons of the analytical results with the numerical results of this work are shown in Fig. 3 and 4. It can be seen that the numerical results employing a finite volume method are very good, even in the beginning of the process, when pressure and displacements show the largest temporal variation. Fig. 4 demonstrates that the finite volume method is also able of solving mechanical problems, since the calculated displacements agree very well with the analytical results.

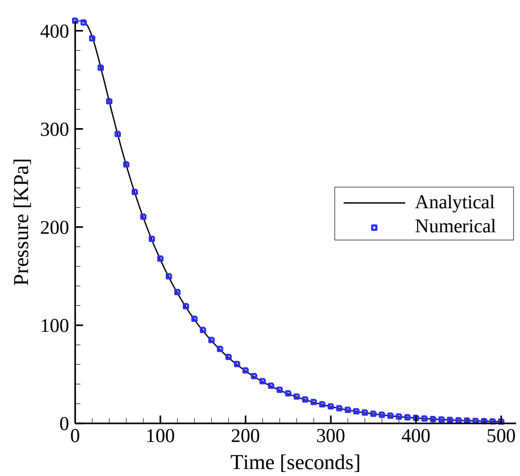


FIGURE 3. Fluid pressure at the bottom of the column.

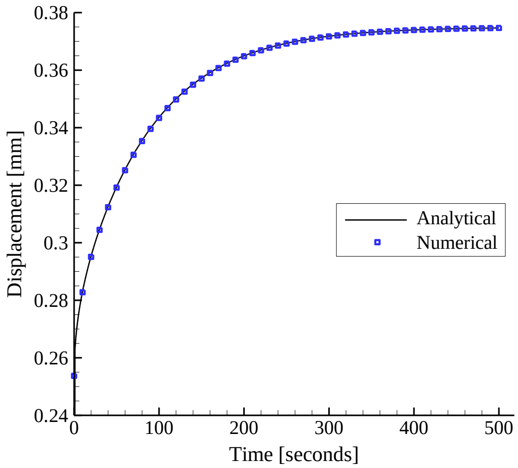


FIGURE 4. Displacements in the y direction at the top.

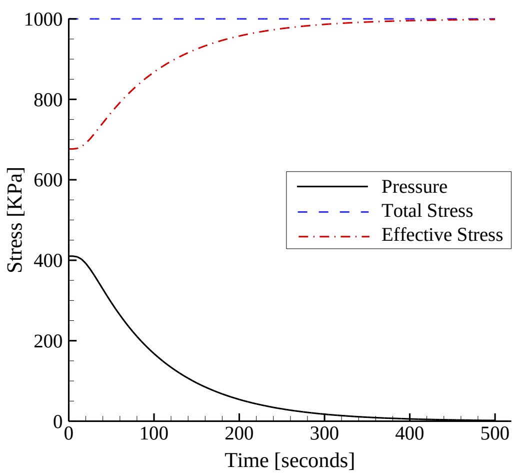
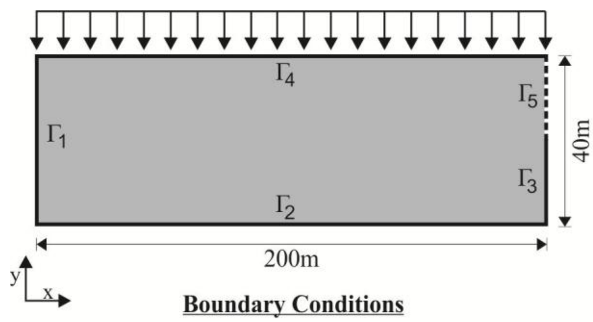


FIGURE 5. Stress equilibrium in the y direction at the bottom

Fig. 5 helps in understanding the physics of the problem during the transient, plotting the stress in the *y*-direction with time. Initially, the stress of 1 MPa is applied, and this load is supported by the fluid and by the porous matrix, since, as mentioned, no fluid is allowed to leave the domain at the time *t=0*. Immediately after, fluid is allowed to leave the domain through the top surface. Therefore, as fluid escapes the domain, fluid pressure decreases and the effective stress in the solid increases, with the corresponding compaction of the porous media. This process goes on until a final equilibrium condition, in which all load is supported by the solid, is reached.

Problem 2. 2D Condition

The second test problem has the loading conditions similar to the previous example, but the fluid is allowed to leave the domain through half of lateral boundary, as depicted in Fig. 6 by a dashed line. This confers to the problem a 2D condition. This problem tries to model a situation encountered in a petroleum reservoir, whereby the well, is connected to the reservoir at the dashed line, while σ_0 is the load applied at the reservoir surface by the layers on top of the reservoir. Therefore, the analysis of this problem aims to determine the integrity of the reservoir media close to the well. The relation between oil production, porosity and rock integrity for each well in the reservoir is of utmost importance. An excess of production that may cause the rock collapse can damage the well permanently. Of course, the analysis of a real problem would require a more realistic model, but the phenomena close to the well is exactly as described and can be solved with the methodology advanced in this work.



Boundary Conditions

Solid:

$\Gamma_1, \Gamma_3, \Gamma_5 \begin{cases} u = 0 \\ \sigma_{xy} = 0 \end{cases}$

$\Gamma_2 \begin{cases} v = 0 \\ \sigma_{xy} = 0 \end{cases}$

$\Gamma_4 \begin{cases} \sigma_{yy} = -\sigma_0 \\ \sigma_{xy} = 0 \end{cases}$

Fluid:

$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \begin{cases} \dot{m} = 0 \\ \text{(waterproof)} \end{cases}$

$\Gamma_5 \left\{ p = p_f \right.$

FIGURE 6. Solution domain and boundary conditios for Problem 2.

The time step employed for the solution is 1 hour, and the results for this problem are reported in Figs. 7 to 9.

Fig. 7 depicts the fluid pressure field and the streamlines for 48 hours of simulation. It can be observed that after this time the whole domain already “felt” the changing in pressure due to the outflow to the well, with the corresponding changing in the porosity.

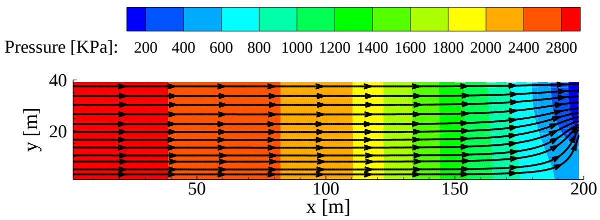


FIGURE 7. Fluid pressure field and streamlines.

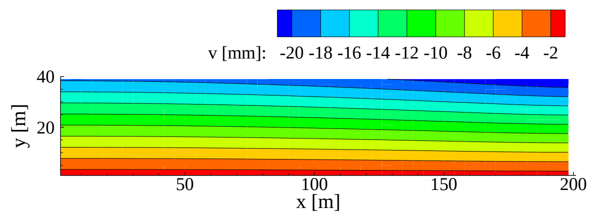


FIGURE 8. Displacement field in the *y* direction.

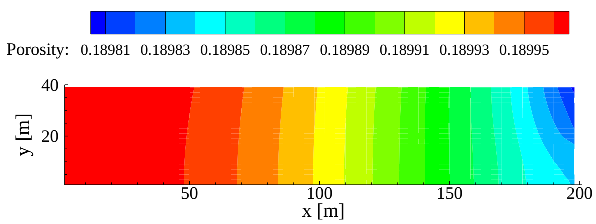


FIGURE 9. Porosity field.

Close to the well, of course, it is where it happen the more severe compaction, with the increase in the effective stress. Fig. 8 shows the displacement and Fig.9 the porosity, results which are consistent with the physics of the problem, which indicates that the porosity has their smallest values close to well, where the strong compaction occurs.

CONCLUSIONS

This paper presented a finite volume method able to solve fluid flow as well as mechanical problems, the so called fluid/structure interaction (FSI) problems, encountered in several engineering applications. It is usual to employ finite volume methods for solving the flow of fluid and finite element method for the stresses filed. As commented along the paper, this requires different grids, variable interpolation and, more severe, in many cases fails to obey the conservation principles, required when physical problems are solved. The methodology advanced herein is general, flexible, physically consistent in its approximation, can be used with unstructured grids and can solve several coupled problems of engineering interest.

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