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ON THE INTRODUCTION OF ARTIFICIAL DISSIPATION IN NUMERICAL METHODS FOR FLUID FLOW PROBLEMS

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SUMMARY

When finite volume methods are used in the solution of fluid flow problems it is normally employed some form of upwinding in order to avoid solution instabilities. In this paper the artificial dissipation introduced when upstream differencing is used is quantified in terms of the cell Peclet number, allowing a better understanding of the role of the numerical diffusion in the context of finite volume methods. It is shown that the upstream scheme is equivalent of using central differencing with the addition of a second order non linear artificial dissipation.

INTRODUCTION

Among the procedures for discretizing the governing partial differential equations of a fluid flow problem, two of them are largely employed by numerical analysts.

In the solution methods known as finite volume or control volume methods the equations in their divergent form are integrated over finite volumes conveniently layed out over the solution domain. As a result of the integration procedure the value of the dependent variable and its spatial derivative are required at the control volume interfaces. As the dependent variable is stored in the center of the control volume, interpolation functions are necessary to evaluate the interface values. The choice of the interpolation functions is of key importance in the solution methodology. Several are the existing possibilities, and each one gives rise to a different numerical solution for the same physical problem, since the interpolation function is intimately connected with the quality of the numerical method. Besides that, the stability of the method, normally of iterative nature, is strongly dependent of the interpolation function.

The simplest choice, the linear interpolation, is not always recommended due to several reasons. First of all, when the cell Reynolds number at the interface is greater than two, the coefficient which connects the dependent variable of the volume under consideration to the adjacent control volume results negative. If iterative methods of solution, like poiny-by-point, line-by-line, etc. are employed, the negative coefficients may cause divergence of the iterative procedure. In addition the central difference scheme (CDS), as is known the procedure where the interpolation is linear, is not dissipative, that is, it does not provide extra mechanisms for the dissipation of errors and perturbations that may occur during the solution, as can be demontrated through a linear stability analysis applied to algebraic operators. As a consequence the convergence rate of the CDS scheme is slow and the converged solutions may present undesirable wiggles. To avoid such a behaviour other differencing schemes are normally used. These schemes are designed in order to maintain the coefficients positive irrespect of the cell Reynolds number, thus allowing the solution to be obtained using the above cited iterative methods for the solution of the linear systems. Many of these schemes have the characteristics of recovering the CDS procedure when the cell Reynolds number is small and the UDS [1] when the cell Reynolds number is large. In the latter case the solution is said to be first order accurate. For intermediate values of the cell Reynolds number the methods use to evaluate the dependent variable at the control volume interface through the exact solution of the one-dimensional diffusion/convection problem. Due to this physical reasoning involving the evaluation of the flow properties at the interfaces these schemes are said to be more physically consistent than the CDS scheme.

In the solution of aerodynamic problems, specially the supersonic ones, the occurence of cell Reynolds number of order of 10000 is not uncommon. In these cases, even if sophisticated interpolation functions, using the cell Reynolds number as the main parameter, are used, the results will be equivalent of using the UDS scheme.

Turning the attention now to the finite difference method, it is seen that the discretization process is conceptually quite different. In these methods the derivatives of first and second order present in the partial differential equation are replaced by their numerical counterpart, normally in central form with the aim of minimizing the truncation errors of the approximation. To promote the stability of the solution, dissipative terms are added to the partial differential equations. Since these terms are chosen to be of fourth order the formal accuracy of the method remains of second order.

In the present work some techniques for introducing artificial dissipation are discussed. Solving the very well known lid-driven cavity problem, firstly it is demonstrated that the UDS scheme is analogous to the CDS scheme with a non-linear second order artificial dissipation. Following, the influences of the second and fourth order dissipative terms with constant coefficients are compared with the influences of the UDS scheme. Also, the results using the UDS scheme and using a fourth order scheme in the solution of a supersonic flow problem are compared. Finally, it is demontrated that the amount of artificial dissipation introduced by the UDS scheme is far more than necessary to guarantee the stability of the solution.

EQUIVALENCE BETWEEN UPWINDING AND ARTIFICIAL DISSIPATION

Recently [2] a segregated formulation in delta form was implemented. The important characteristic of this formulation is the fact that the unknows are temporal variations of the conserved properties. This approach is usually employed in the solution of compressible flows using finite-difference techniques with the linear systems originated from each conservation equation solved simultaneously. The delta form has several advantages and no drawbacks when compared with conventional segregated formulations. Results pointing out these advantages are reported in [2] and [3].

For the sake of simplicity, consider the laminar, two dimensional, incompressible flow with constant properties. The discretized equations written for a generic scalar ϕ in delta form are [2]

$$a_p\Delta\phi_P - a_e\Delta\phi_E - a_w\Delta\phi_W - a_n\Delta\phi_N - a_s\Delta\phi_S = RHS \quad (1)$$

where

$$RHS = -\left[J_e - J_w + J_n - J_s\right] - L\,[P^\phi]^t \quad (2)$$

In Eq.(2) J denotes the convective and diffusive flux of ϕ at the interfaces of the elemental control volume. The RHS corresponds to the discretization of the steady state part of the partial differential equation. Writing ϕ and its derivative for the east face, for example, of a control volume by

$$\phi_e = (1/2+\alpha)\phi_P + (1/2-\alpha)\phi_E \quad ; \quad \left.\frac{\partial\phi}{\partial x}\right|_e = \beta\,\frac{\phi_E - \phi_P}{\Delta x} \quad (3)$$

the Eq.(2) can be put in the following form

$$RHS = -a_p^*\phi_P^t + a_e^*\phi_E^t + a_w^*\phi_W^t + a_n^*\phi_N^t + a_s^*\phi_S^t - L[P^\phi]^t\Delta V \quad (4)$$

Note that the term RHS depends only on known values and the coefficients a_p^*, a_e^*,... are not trully coefficients of a linear set of equations like the coefficients a_p, a_e, a_w,... present in the left hand side of Eq.(1). Note also that all the schemes based on one-dimensional interpolations are recovered with the use of Eq.(3). If the CDS scheme ($\alpha = 0$, $\beta = 1.0$) or UDS ($|\alpha| = 0.5$, $\beta = 1.0$) is employed, the a_e^* coefficient results, respectively

$$[a_e^*]_{CDS} = -\dot{M}_e/2 + D_e \quad (5)$$

or

$$[a_e^*]_{UDS} = (|\dot{M}_e| - \dot{M}_e)/2 + D_e \quad (6)$$

where

$$D_e = \Gamma^\phi\Delta y/\Delta x \quad ; \quad \dot{M}_e = \rho u_e\Delta y \quad (7)$$

Eq.(6), after some algebraic manipulations can be put in the form

$$[a_e^*]_{UDS} = -\dot{M}_e/2 + \Gamma^\phi\left[1 + |Pe|_e/2\right]\frac{\Delta y}{\Delta x} \quad (8)$$

where Pe is the Peclet number (Reynolds number in the case of momentum conservation) defined by

$$Pe|_e = \rho u_e\Delta x/\Gamma^\phi \quad (9)$$

If an effective diffusion coefficient is defined as

$$\Gamma^\phi_{eff} = \Gamma^\phi[1 + |Pe|_e/2] \quad (10)$$

Eq.(8) results

$$[a_e^*]_{UDS} = -\dot{M}_e/2 + D_{eff,e} \quad (11)$$

where

$$D_{eff} = \Gamma_{eff}\Delta y/\Delta x \quad (12)$$

Eq.(11) demonstrates that the coefficient obtained using the UDS approach is equivalent of using the CDS approach with an effective diffusion coefficient given by Eq.(10). Numerical experiments show that the solutions obtained with the UDS approach applied through Eq.(6) or Eq.(10) are identical. It is important to remember that in the solution of high speed flows the effective coefficient is much larger than the molecular coefficient.

As a conclusion, the UDS scheme is equivalent to the CDS scheme with a non-linear second-order dissipative term added to the discretized partial differential equations. Relations similar to Eq.(10) can be obtained for other schemes as exponential [1], WUDS [8], etc. Following, the consequences of the UDS scheme when applied to a specific problem are discussed. In fact the analysis is valid for all schemes represented by Eq.(3) but with a little more difficulty of identification.

<u>Consequences of using the UDS Scheme</u>. Solving the lid-driven cavity problem with Reynolds number equal to 1000, where the characteristic dimension is taken as the side of the square cavity and the characteristic velocity the velocity of the moving wall, using the CDS scheme with a 10X10 grid, the maximum nodal velocity, dimensionless with respect to the wall velocity, is 0.3732. Using the UDS scheme the same velocity reduces its value to 0.3098. Of course, the difference is not seen only in the maximum velocity. All fields of u, v and P show considerable differences. It is clear that with a very fine grid both results should be coincident, since the artificial diffusion introduced with the UDS diminishes with the cell size and both schemes reduce to the same.

The understanding that the UDS scheme is equivalent to the CDS with an artificial dissipation helps to explain why the maximum velocity is smaller when the UDS scheme is applied. The computation of the artificial dissipation coefficient, according to Eq.(10), at the interfaces of the control volumes will show that this coefficient is up to 15 times greater than the molecular coefficient. Using the UDS the artificial diffusion will only be zero at the walls of the cavity since they are impermeable, reducing the cell Reynolds number to zero. Therefore, the solution using UDS is physically equivalent of solving the problem of the lid-driven cavity with a fluid with different molecular viscosity, smaller at the wall. Since the movement of the interior fluid is induced by the stress occuring at the wall, it is clear why the maximum velocity is smaller using UDS.

Another very important point is to answer when to use the CDS or the UDS scheme. Obviously, the answer depends on the definition of criteria to be followed:

i) If the computer effort is taken into account the difference is considerable. 87 iterations for the UDS scheme against 212 for the CDS (keeping all other parameters constant). This behaviour is, as already discussed, due to the non-dissipative character of the CDS scheme.

ii) If the solution quality is the prime parameter, the solution obtained using CDS is superior, in the present test, as demonstrated by a grid refinement study.

In fact, the best way to answer this question would be to compare the solutions obtained with the same computer effort using both schemes. For the same computer effort the UDS scheme would allow the use of more refined grid. This test was not implemented because, among another reasons, the conclusion depends on the specific problem under analysis and so, can not be generalized.

ARTIFICIAL DISSIPATION WITH CONSTANT COEFFICIENTS

In the majority of the methods applied in the solutions of aerodynamic problems the discretization is done through finite-differences. To promote stability fourth-order dissipative terms with constant coefficients are introduced. Besides the fact that these terms are not expected to damage the solution it is illustrative to investigate the influences of second-order terms (like UDS) on the solution. To this end the lid-driven cavity problem is again considered.

A second-order dissipation can be introduced simply adding to the RHS term given by Eq.(4) a term $D_e^{(2)}$ of the type

$$D_e^{(2)} = \omega_e(\phi_N - 2\phi_P + \phi_S) + \omega_e(\phi_E - 2\phi_P + \phi_W) \quad (13)$$

where ω_e is the artificial dissipation coefficient and the subscript indicates that it actuates in the explicit part of the discretized governing equations. If the problem is solved with the RHS term evaluated using CDS and with $\omega_e = 0.003$ the solution is obtained in 80 iterations against 212 of the original CDS. The maximum nodal velocity, however, increased to 0.5601, is much larger than the ones reported above. It is easy to verify that the solution via CDS with ω_e = 0.003 is identical to the solution via CDS with ω_e = 0.0 and

$$Re_L = \frac{1}{0.001 + 0.003} = 250 \quad (14)$$

In fact, to add the dissipative terms given by Eq.(13) is equivalent of increasing the molecular viscosity of the fluid. Note that the dissipative coefficient introduced is three times the molecular coefficient, while in the solution via UDS (with dissipative terms of second-order but non-linear) this coefficient, in the problem analysed, varies from zero to 15 times the molecular coefficient. Futhermore, note that the same result obtained with ω_e = 0.003 could be obtained simply placing $\alpha = 0$ and $\beta = 4$ in Eq.(3).

Fourth-order dissipation can be introduced if it is added to the RHS given by Eq.(4) terms like

$$D_e^{(4)} = \omega_e(\phi_{NN} - 4\phi_N + 6\phi_P - 4\phi_S + \phi_{SS}) + $$

$$+ \omega_e(\phi_{EE} - 4\phi_E + 6\phi_P - 4\phi_W + \phi_{WW}) \quad (15)$$

The number of iterations decreases with ω_e up to ω_e = 0.001, when 145 iterations are needed, resulting in a maximum velocity of 0.51055. However, for ω_e larger than 0.001 the procedure diverges. The linear stability analysis applied to the methods of simultaneous solution show that the ω_e coefficient is upper bounded [4] and that this limit can be extended if second-order artificial dissipation is added to the left hand side of the equation, that is, to the implicit part of the equation.

An implicit second-order dissipation can be introduced if it is added to the left hand side a term of the type

$$D_i^{(2)} = - \omega_i \{[(\Delta\phi)_E - 2(\Delta\phi)_P + (\Delta\phi)_W] + $$

$$[(\Delta\phi)_N - 2(\Delta\phi)_P + (\Delta\phi)_S]\} \quad (16)$$

As a consequence the coefficients a_e, a_w, a_n e a_s will be augmented of ω_i while the a_p coefficient will be augmented of four times ω_i. As expected, for $\omega_i = 0.004$, the limit of ω_e, of 0.001 before, is extended to 0.0032. There is no, however, improvements in the convergence rate and the solution is poorer than the one obtained via CDS. The introduction of implicit dissipation with coefficient equal to 0.004, for example, is equivalent to adopt $\beta = 5$ in Eq.(3). Recall that the steady state solution does not depend on the coefficient evaluation when the formulation is in delta form.

A FINITE-VOLUME SCHEME FOR THE SIMULTANEOUS METHODS

In the methods where the discretization of the equations is done through central finite-differencing the convective term $-\partial(\rho uu)/\partial x$, for example, is approximated by

$$\frac{\partial}{\partial x}(\rho uu) \cong [(\rho uu)_E - (\rho uu)_W]/(2\Delta x) \quad (17)$$

where the E and W locations are shown in Fig. 1.

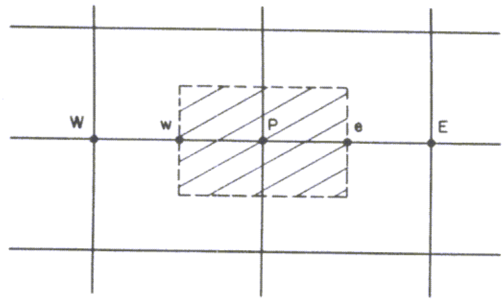

Fig. 1 A control volume for the B&W scheme.

In the other hand, one can imagine a control volume surrounding P and evaluate the same derivative through the following expression

$$\frac{\partial}{\partial x}(\rho uu) = [(\rho uu)_e - (\rho uu)_w]/\Delta x \quad (18)$$

Consider now the first part in the interior of the brackets. The term $(\rho uu)_e$ can be split in the product of a mass flux by a momentum flux, i.e,

$$(\rho uu)_e = (\rho u)_e (u)_e \quad (19)$$

Like in the segregated finite volume method, the mass flux can be estimated through the averaging process given by

$$(\rho u)_e = [(\rho u)_P + (\rho u)_E]/2 \quad (20)$$

The u velocity at east face can be evaluated by several schemes. In the next section results will be reported where the value of all the dependent variables at the interfaces were evaluated by the UDS scheme.

ANALYSIS OF THE INFLUENCES OF SEVERAL DISSIPATIVE TERMS IN THE SOLUTION OF A COMPRESSIBLE FLOW PROBLEM

The several schemes to intoduce artificial dissipation reported previously were implemented for the solution of the axisymmetric flow of air with free-stream Mach number equal to 1.5 against an hemisphere-cylinder shown in the insert of Fig. 2. The figure shows curves of C_P along the symmetry line obtained using simultaneous and segregated methods for several types of artificial dissipation. The solutions should predict the presence of a shock located approximately at $x/R = -0.6$ [7]. The curve (a) was obtained using the scheme proposed by Beam and Warming [4] with fourth-order dissipation with constant coefficients. The solution shows a smeared shock due to the coarse grid employed. The non-physical prediction of negative C_p before the shock wave is characteristic of the fourth-order schemes. The (b) curve was obtained using the segregated method due to Maliska and Silva [6], implemented in nonorthogonal coordinates with the coefficients in all equations, included mass conservation, evaluated via CDS. The oscillation in the solution is not surprising since it is a characteristic of solutions free of extra dissipative terms in high Reynolds number flows. The shock, however, is well captured and located in only two cells. The curve (c) was obtained by the same scheme of curve (b) but with the CDS replaced by UDS. It is noted that there is no irrealistic values nor oscilations of C_p but the shock is strongly attenuated. If in the Beam and Warming scheme, the explicit part of the equations is evaluated using the UDS scheme, in the finite-volume approach proposed in the previous section of the present work, the C_p distribution (not shown in Fig. 2) is very similar to curve (c). This fact clearly demonstrates that the differences in the solutions must be credited to the way in which artificial dissipation is introduced. It should be also mentioned that in this case it is no longer needed to apply artificial dissipation to the simultaneous scheme through expressions like Eq.(15). Finally, it was tested a different scheme, were the number 2 appearing dividing the Peclet number in Eq.(10) is substituted by 20, giving rise to a dissipative scheme, non-linear, with ten times less dissipation than the UDS. The result is shown in curve (d). The shock is well captured, as in the case of the fourth-order scheme but without the negative values of C_p and maintaining the same convergence rate of the UDS scheme.

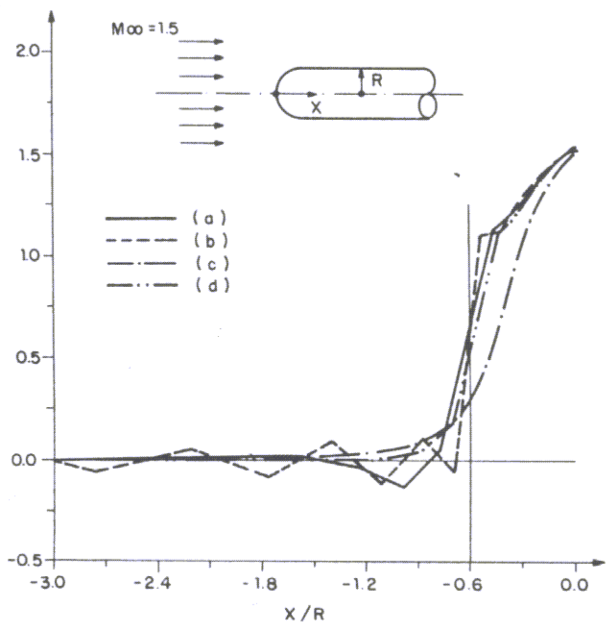

Fig. 2 The C_p curve obtained with a)the original Beam & Warming scheme and fourth order dissipation; b) the segregated FV scheme and CDS; c) the segregated FV scheme and UDS, and d) the segregated FV scheme and reduced artificial dissipation.

CONCLUDING REMARKS

The outcome of the present work reveals that:

a) The majority of the interpolation schemes employed in the finite-volume methods can be put in the form of a CDS scheme with the addition of non-linear second-order dissipation;

b) These schemes, which reduce to the UDS scheme for high cell Peclet numbers, introduce much more dissipative effects than necessary to promote stability and to eliminate spurious oscillations in the solution;

c) The differences observed in the solutions obtained via the segregated methods, as the one proposed in [6], and the simultaneous methods, as the one proposed in [4], are due to the way in which artificial dissipation is introduced;

d) The second-order dissipation with constant coefficients can be implemented in the finite-volume method by simply using values of β in Eq.(3) larger than unity; and

e) It is recommended that new schemes of introducing controlled artificial dissipation be devised in the context of the segregated methods. The study should be based on stability analysis, as is done in the context of simultaneous solution methods, and not on the maintenance of the positivity of the coefficients which, as demonstrated in [2], does not influence the stability of the iterative procedure.

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