A FULLY-CONSERVATIVE FINITE VOLUME FORMULATION FOR COUPLED PORO-ELASTIC PROBLEMS

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Solid mechanics is a research field that deals with the mechanical behavior of a wide variety of materials undergoing external loads. Among the various types of solids, porous materials, for instance, can be found in applications such as soil and rock mechanics, biomechanics, ceramics, etc. These applications are studied in the field of poromechanics, which is a specific branch of the solid mechanics that considers all types of porous materials. An important characteristic of such materials is that they contain a network of interconnected pore channels saturated with a fluid. In most situations the mechanical behavior of the porous matrix and the fluid flow through the pore channels are two tightly coupled phenomena interfering with each other. When the fluid moves from one region to another in the porous matrix it changes the pressure field inside the pore channels, which is perceived by the porous matrix as a force imbalance. As a consequence, the porous matrix tends to deform in order to find a new configuration of stress equilibrium. When the porous matrix deforms, the pore channels are also modified, which directly affects the fluid flow and the pore pressure field. It is clear then that a fluid flow model and a structural model must be considered in order to solve this coupled phenomenon.

The basis of the theory that describes coupled poromechanics has been established by Terzaghi (1923) [1], where the effective stress principle has been presented. According to this principle, the effective stresses $\boldsymbol{\sigma}'$ acting on the solid porous matrix is balanced by the pore pressure p and the total stress tensor $\boldsymbol{\sigma}$ externally applied to the system, that is:

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \alpha p \boldsymbol{I} \tag{1}$$

where α is the Biot's coefficient and $\mathbf{I}$ is the second-order identity tensor. Almost 20 years later, Biot (1941) [2] generalized this theory to three-dimensions and it has become known as Biot's consolidation theory. In this theory, the governing equations are the mass conservation equation for deformed porous media and the stress equilibrium equations for a porous matrix. Considering only small strains and linear behavior for the porous matrix, the stress equilibrium equations can be written as:

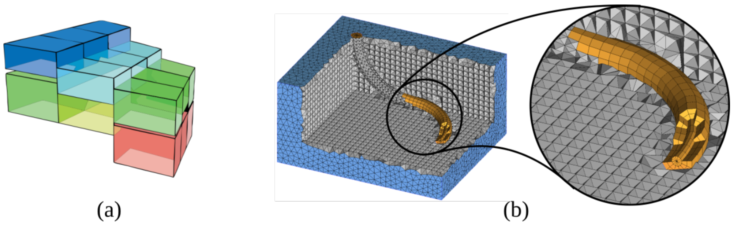
$$\nabla_s \cdot (\mathbb{C}\nabla_s \mathbf{u} - \alpha p \mathbf{i}) + \rho \mathbf{g} = 0 \tag{2}$$

with $\mathbb{C}$ being the fourth-order tensor written in Voigt notation, ∇_s being the symmetric nabla operator and $\mathbf{i}$ being the Voigt represetation of $\mathbf{I}$. Finally, the mass conservation equations in this case reads,

$$\frac{1}{M}\frac{\partial p}{\partial t} + \nabla \cdot \left(\mathbf{v}^f + \alpha \mathbf{v}^s\right) = q \tag{3}$$

with the Biot's modulus given by a combination of the solid and fluid compressibilities (c_s and c_f, respectively), porosity ϕ and α, that is, $M = [\phi c_f + (\alpha - \phi) c_s]^{-1}$. Additionally, the seepage velocity (Darcy velocity) and the solid velocity are respectively given by:

$$\mathbf{v} = -\frac{\mathbf{k}}{\mu} \cdot (\nabla p - \rho_f \mathbf{g}) \quad \text{and} \quad \mathbf{v}^s = \frac{\partial \mathbf{u}}{\partial t} \tag{4}$$



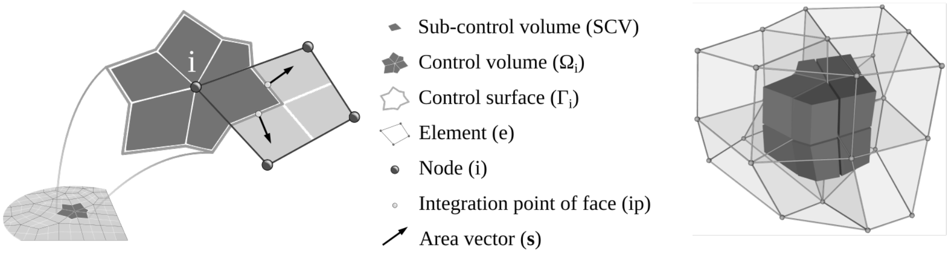
(a) (b)

Figure 1: (a) Corner-point and (b) unstructured grids.

in which **k**, μ, ρ_f, **g** and **u** stand for the absolute permeability tensor, fluid viscosity and density, gravitational acceleration vector and the displacement vector, respectively.

For real applications, the system of coupled partial differential equations composed of Equations (3) and (2) must be solved by numerical techniques. In the groundwater community, the most common approach is to apply the Galerkin Finite Element Method (FEM) for discretizing both fluid flow and structural models. The use of FEM for solving coupled geomechanics is probably because of historical reasons, as problems involving solid mechanics have always been solved by FEM. Although this method presents the advantage of being applied to unstructured grids (see Figure 1b), thus providing great geometrical flexibility, it does not ensure local mass conservation, which is an important characteristic specially for multiphse flows. In reservoir simulators, for instance, where multiphase flows are the main mathematical models considered, the Finite Volume Method (FVM) is the most common choice. The reason for employing the FVM in reservoir simulators is because its basic premisse is to ensure local conservation in every control volume of the grid. In this context, another common approach for solving coupled geomechanics is to solve the fluid flow model in a resevoir simulator with the FVM, and then solve the geomechanical model in a separate FEM software. There are a number of drawbacks in this approach that deserves further discussion. First, reservoir simulators are usually applied to corner-point grids, as the one depicted in Figure 1a, where the variables are stored at the grid block centroids. In the FEM software, unstructured grids are usually employed, with the variables stored at the grid nodes. This situation requires the interpolation of variables between two different grids, which represents an extra source of numerical errors and additional computational cost. Moreover, synchronizing two different softwares and managing the traffic of information between the two of them can be a cumbersome task. In order to avoid these drawbacks, a number of researchers have been proposing unified methodologies for solving both geomechanical and fluid flow models. In the FEM community, a number of works in this direction can be mentioned ([3, 4, 5] and many others). On the other hand, a few important attempts have being proposed for solving coupled geomechanics in a unified finite volume formulation. For instance, Shaw & Stone (2005) [6] solved linear poroelasticity in unstructured cell-centered grids, although emphazis has been placed on corner-point grids. Later on, dal Pizzol and Maliska (2013) [7] presented a finite volume formulation for coupled geomechanics in Cartesian staggered grids for two-dimensional problems. Important advances on cell-centered finite volume formulations were also developed in [8, 9] for two-dimensional unstructured grids.

The present work proposes the solution of coupled geomechanics by employing the Element-based Finite Volume Method (EbFVM) for discretizing the partial differential equations of both fluid flow and geomechanical models. As a finite volume method, the EbFVM ensures mass and momentum (force) conservation for each control volume of the grid, which is an important feature specially for fluid flows. Moreover, the EbFVM provides great geometrical flexibility as it is naturally applied to unstructured grids. In this work, three-dimensional unstructured grids composed of tetrahedra, hexahedra, prisms and pyramids are employed. These types of grids are of particular interest for building radial grids around wells in order to better capture the radial flow pattens in this region (see Figure 1b). In the EbFVM, the control volumes are built around the nodes of the grid, therefore the variables of the problem (p and **u**, in this case) are stored at the grid nodes, characterizing a cell-vertex method. As shown in Figure 2a, each element of the grid is subdivided into sub-elements, or sub-control volumes, associated to each element vertex. The control volume is then built by the union of all sub-elements sharing a common node. Figure 2b shows a control volume built around a node of a three-dimensional grid. Each control volume Ω_i is bounded by a control surface Γ_i composed of faces identified by one integration point, ip, on its centroid



(a) (b)

Figure 2: Geometrical entities for (a) two-dimensional grid and (b) three-dimensional grid.

and an area vector, **s**, point outwards the control volume.

The discretized mass and stress equilibrium equations are obtained by integrating Equations (3) and (2) in each control volume and applying the divergence theorem. The resulting surface integrals over Γ_i are then evaluated at the integration points of the control volume. This means that mass fluxes and forces are computed at the control volume surfaces, which is precisely what ensures mass and momentum (force) conservation. When the algebraic representation of Equations (3) and (2) are grouped together, the following linear system is obtained:

$$\begin{bmatrix} -K & L \\ \dfrac{Q}{\Delta t} & \dfrac{A}{\Delta t} - H \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{b}^{\mathbf{u}} \\ \mathbf{b}^{p} \end{bmatrix} \tag{5}$$

where the block matrices K and L accounts for the effective stresses and the pore pressures acting on control volumes' surfaces. The second block-line of Equation (5) contains the mass conservation equations, where matrices A, H and Q represent the accumulation terms, the mass fluxes due to the seepage velocity and the mass fluxes due to the solid movement. The linear system of Equation (5) is solved in a monolithic way by an LU decomposition.

The proposed methodology is first validated against the well known Mandel's problem, where a rock slab is compressed in vertical direction and the lateral boundaries are fully permeable, as depicted in Figure 3. In this problem, the poroelastic equations cannot be decoupled as in the one-dimensional poroelastic column of Terzaghi, which makes it a suitable test case for assessing the proposed formulation. The problem has been solved with grids composed of four types of elements: hexahedra, tetrahedra, prisms and pyramids. As it can be seen, good agreement with the analytical solution is obtained for all types of grids.

The final problem intends to reproduce a water withdrawal from a 12 meters aquifer composed of sand. The aquifer is trapped between two layers of silty clay with low permeability. The whole structure consists of a cylinder with 250 meters radius and 50 meters height. A vertical well with prescribed

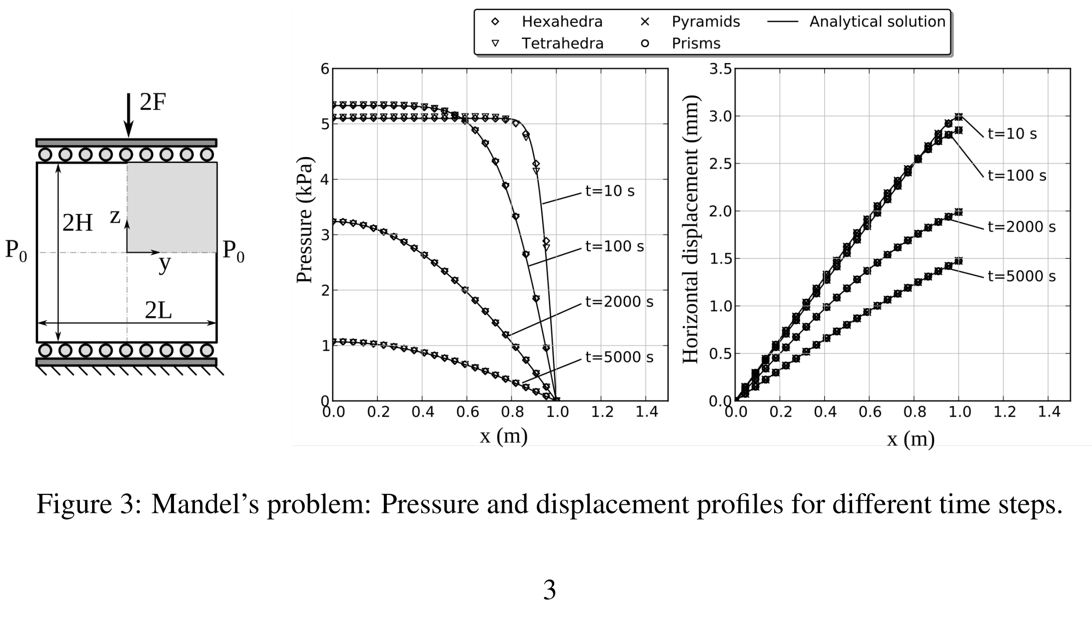


Figure 3: Mandel's problem: Pressure and displacement profiles for different time steps.

constant pressure is placed at the center of the structure. Due to symmetry, only a quarter of the geometry is considered, as depicted in Figure 4. The left side of this figure shows the pressure and vertical displacement fields. The graphic in the middle shows the pressure profile along the vertical center line of the structure. The well is placed between $z = 27,5$ and $z = 40$ meters, where pressure is constant. It is interesting to notice the positive values of pressure that establishes in the adjacent aquitards. This is known as the Noordbergum effect and it's an evidence of the coupling between fluid flow and geomechanics. The rightmost graphic of Figure 4 shows the vertical displacement in the radial direction. After 250 days, the maximum subsidence observed is of 40 mm right above the well.

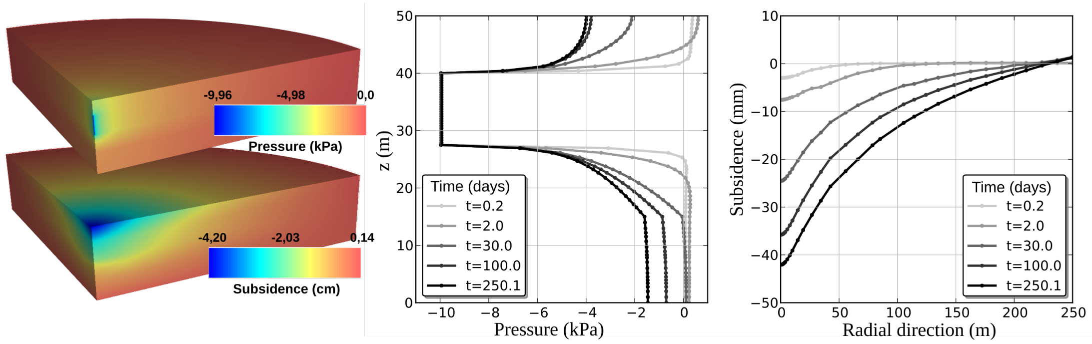


Figure 4: Groundwater withdrawal.

In this work the EbFVM has been used for solving both physical models involved in geoemechanics: the fluid flow and geomechanical model. This a promising alternative for solving coupled geomechanics for two main reasons. Since it is a fully conservative method, it is able to accurately solve multiphase flows in porous media. Moreover, the momentum equation is also satisfied for each control volume of the grid. The second reason is because it is able to handle unstructured grids composed of different types of elements. This allows for the use of radial grids in the near-well region in order to better capture the flow patterns in the vicinity of the well. To the knowledge of the authors, there is no other numerical scheme that present all this features together.

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